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Inhomogeneous Similarity Solutions of the Boltzmann Equation with Confining External Forces

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The Nikolskii transform makes it possible to construct inhomogeneous solutions of the Boltzmann equation from homogeneous ones. These solutions correspond to a gas in expansion, but if we introduce external forces, they can relax toward absolute Maxwellians. This property holds independently of the assumed intermolecular inverse power force. Consequently, for Maxwell molecules and from energy-dependent homogeneous distributions, we construct effectively a class of inhomogeneous similarity distributions with Maxwellian equilibrium relaxation. We review and investigate again the homogeneous distributions which can be written in closed form, for instance, we show that an elliptic exact solution proposed some years ago violates positivity. For Maxwell interaction with singular cross sections, we numerically construct inhomogeneous distributions having Maxwellian equilibrium states and study the Tjon overshoot effect. We show that both the sign and the time decrease of the external force as well as the microscopic model of the cross section contribute to the asymptotic behavior of the distribution. These inhomogeneous similarity solutions include a class of distributions that asymptotically oscillate between different Maxwellians. Two classes of external forces are considered: linear spatial-dependent forces or linear velocity-dependent forces plus source term.

KEY WORDS: Boltzmann equation; kinetic theory; nonlinear equations.

1. INTRODUCTION

More than 20 years ago, for the Boltzmann equation without external forces, Nikolskii[1] discovered a transform that makes it possible to build up a class of inhomogeneous solutions $f(\mathbf{v}, \mathbf{x}, t)$ ($\mathbf{v}$ is the velocity, $\mathbf{x}$ the space coordinate, and t the time) from homogeneous ones $F(\boldsymbol{\eta}, \tau)$ ($\boldsymbol{\eta}$ is the

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velocity, τ that time). Through this transform F and f become identical, f being an inhomogeneous distribution; $\boldsymbol{\eta}(\mathbf{v}, \mathbf{x}, t)$ and $\tau(t)$ are well-defined functions. The result holds independently of the intermolecular inverse power law (although some minor differences occur[1]). The drawback is that when t goes to infinity, the inhomogeneous distribution, the temperature T, and the local density ρ go to zero, leading to the physical interpretation of a gas in expansion. When Bobylev[2] discovered his exact homogeneous distribution, he applied the Nikolskii transform to it and found the first explicit known inhomogeneous distribution with, unfortunately, the above-mentioned relaxation toward zero.

The need for explicit three-dimensional ($d = 3$) inhomogeneous distributions relaxing toward absolute Maxwellian equilibrium distributions has remained (only for the $d = 1$ Kac model with momentum conservation dropped has such a kind of relaxation been found[3]). Recently[4] this difficulty was overcome for Maxwell molecules by introducing external forces linear in the space variable (harmonic potentials) or in the velocity variable (with, in addition, a source term). Explicit inhomogeneous similarity solutions with the same analytic structure as the Bobylev–Krook–Wu[5] homogeneous distribution were obtained, and the equilibrium distributions were absolute Maxwellians. So these forces act like confining external forces because the gas, instead of expanding in space, remains in a Maxwellian equilibrium state.

Here we want to clarify and extend the result quoted above, keeping these external confining forces. First we introduce other intermolecular forces besides those leading to Maxwell particle interactions. Second, starting with the whole class of homogeneous energy-dependent distributions, we extend the class of exact inhomogeneous similarity distributions relaxing toward absolute Maxwellians. In order to roughly understand the results, we can briefly say that for large time the distributions are similar to local Maxwellians f_{LM}

$$f_{\mathrm{LM}} = \rho(2\pi T)^{-d/2} \exp(-\mathbf{c}^2/2T) \tag{1.1}$$

where $\mathbf{c}$ is the peculiar velocity, $\mathbf{c} = \mathbf{v} - \langle \mathbf{v} \rangle$. The mean velocity $\langle \mathbf{v} \rangle$ depends on $\mathbf{x}$ and t, while ρ, the local density, and T, the temperature, are only t dependent. If the external forces are absent, then the temperature T goes to zero, $f_{\mathrm{LM}} \to 0$, whereas if the confining forces are present, we can find $T \to \mathrm{const}$ and

$$f_{\mathrm{LM}} \to f_{\mathrm{Abs\ Max}} = \mathrm{const} \times \exp(-\mathrm{const} \times \mathbf{c}^2)$$

We can also obtain relaxation toward oscillating Maxwellians if T oscillates when t is growing. For the inhomogeneous similarity solutions we must at least require $\bar{\mathscr{L}}(\mathbf{c}^2/2T)=0$, where $\bar{\mathscr{L}}$ is the differential part of the linear Boltzmann operator $\mathscr{L}$. There exists a connection between this condition and $\mathscr{L}(f_{\mathrm{LM}})=0$, the condition leading to the determination of the solutions of the linear part of the B.E. alone. Since the temperature is determined by the above condition and T is responsible for the asymptotic relaxation, the existence of Maxwellian equilibrium states is independent of the intermolecular force laws. We choose external forces that are spatially dependent in Sections 2–4 and that are linearly velocity dependent with source term in Section 5.

In Section 2, we consider the $d>1$ dimensional B.E. with arbitrary repelling intermolecular forces, and, introducing spatial external forces, we generalize the Nikolskii transform. The dependences of T, ρ, and $\langle \mathbf{v} \rangle$ are obtained from the external forces, so that the different possibilities of relaxation are independent of the intermolecular forces. However, only for Maxwell molecule intermolecular forces and spatially homogeneous distributions with even velocity dependence is the explicit construction of Boltzmann distributions presently known, so that is Sections 3 and 4 we restrict our attention to these cases.

In Section 3 we recall and discuss the well-known homogeneous energy-dependent formalism.[2,5–7),2] We write down the Laguerre expansions of the distributions, the Laguerre moments being recursively determined. The interest is that all these homogeneous solutions can also be viewed as inhomogeneous similarity distributions with absolute Maxwellian relaxations. We also reconsider the nonlinear partial differential equations (NLPDE)[(5,8,9)3] as well as a nonlinear integrodifferential equations[(10)] obtained some years ago for the generating functionals of the power moments. We discuss the possible exact solutions and obtain new "solitons" and "bisolitons." We focus our attention on the homogeneous similarity solutions and, for instance, prove rigorously that a Weierstrassian elliptic solution proposed some time ago by Ernst[(7)] as a new Boltzmann distribution must be rejected because it violates positivity.

In Section 4 we perform some numerical calculations for the inhomogeneous similarity distributions associated with temperatures going to constants or oscillating. In the first case we study the Tjon overshoot effect[(11)] for the reduced distribution (ratio of the distribution to its asymptotic absolute Maxwellian). We take the opportunity to compare the effect in both homogeneous and inhomogeneous formalisms. We find new

[2] See Refs. 7 for reviews and additional references.

[3] Reference 9 discussed homogeneous solutions outside the usual L^2 space spanned by the Laguerre polynomials. To these homogeneous distributions also correspond also a class of inhomogeneous similarity distributions having the relaxation properties studied in the present paper.

features: the sign and the decrease of the external force as well as the microscopic cross section are important. All the numerical calculations are performed with the true Maxwell inverse fourth power interaction. This means that we must take into account a singular differential cross section.

In Section 5 we study external forces that depend linearly on velocity plus source term. We find two possibilities, depending on whether $\bar{\mathscr{L}}$, the differential part of the linear Boltzmann operator, is identical to $\mathscr{L}$ or not. In the second case, the asymptotic behavior of the distribution contains a pure time factor, leading to asymptotic behavior incompatible with that of the Gaussian part of the Maxwellian. Singular asymptotic behavior, such as a delta distribution, can occur. We disregard this uninteresting case, since we seek absolute Maxwellians.

For the first possibility, as for the spatial harmonic potential, the asymptotic behavior of the inhomogeneous distribution can be oscillating as well as absolute Maxwellian. Only the differential relations between temperature and the time-dependent part of the outside force are different. This means that if we start with a given temperature, we can associate both a spatially dependent force model and a velocity-dependent force plus source terms. Consequently, all the results of Sections 3 and 4 can be applied to this problem. In particular, the numerical distributions constructed in Section 4 for spatial forces can be reinterpreted in terms of velocity-dependent forces plus source term.

2. NIKOLSKII TRANSFORM IN THE PRESENCE OF EXTERNAL FORCES

2.1. Inhomogeneous versus Homogeneous Formalisms for Spatial Forces

We write down the B.E. for inhomogeneous d-dimensional distributions $f(\mathbf{v}, \mathbf{x}, t)$ ($\mathbf{x}$ and $\mathbf{v}$ are $d>1$ dimensional vectors) in the presence of spatially dependent external forces $\mathbf{A}(\mathbf{x}, t)$. We assume intermolecular forces with inverse power law p:

$$\mathscr{L}f(\mathbf{v}, \mathbf{x}, t) = \mu \operatorname{Col} f(\mathbf{v}, \mathbf{x}, t), \qquad \mathscr{L} = \partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}} + \partial_{\mathbf{v}}\mathbf{A}$$

$$\operatorname{Col} f = S_d^{-1} \int d\Omega_d\, d\mathbf{w}\, \sigma^{(d)}(x)\, g^{1-2(d-1)/(p-1)}[f(\mathbf{w}')\,f(\mathbf{v}') - f(\mathbf{v})\,f(\mathbf{w})] \tag{2.1}$$

μ is a constant, $S_d = \int d\Omega_d$, $\mathbf{g} = \mathbf{v} - \mathbf{w}$ is the reltive velocity, $|\mathbf{g}| = g = |\mathbf{g}'| = |\mathbf{v}' - \mathbf{w}|$; $d\Omega_d$ is the d-dimensional infinitesimal solid angle

$$d\Omega_d = (\sin\chi)^{d-2}(\sin\varepsilon)^{d-3}(\sin\varepsilon_1)^{d-4}\cdots\sin\varepsilon_{d-4}\, d\chi\, d\varepsilon_1\cdots d\varepsilon_{d-3}$$

expressed as a function of the $d-2$ polar angles $\chi, \varepsilon, \varepsilon_1,..., \varepsilon_{d-3} \in [0, \pi]$ and of the azimuthal angle $\varepsilon_{d-4} \in [0, 2\pi]$ of $(\mathbf{g}, \mathbf{g}')$ $\sigma^{(d)}(\chi)$ is the cross section and θ the angle between $\mathbf{v}$ and $\mathbf{w}$. We assume momentum and energy conservation: $\mathbf{v}+\mathbf{w}=\mathbf{v}'+\mathbf{w}'$ and $\mathbf{v}^2+\mathbf{w}^2=\mathbf{v}'^2+\mathbf{w}'^2$, leading to

$$\begin{pmatrix} \mathbf{v}'^2 \\ \mathbf{w}'^2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}^2 \\ \mathbf{w}^2 \end{pmatrix} + \left[(\mathbf{w}^2-\mathbf{v}^2)\sin^2\frac{\chi}{2} + |\mathbf{v}|\,|\mathbf{w}| \sin\chi \sin\theta \cos\varepsilon \right] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2.1'}$$

The main assumption is that f is an inhomogeneous similarity solution identical to a homogeneous distribution F with $\boldsymbol{\eta}$ and τ as velocity and time variables:

$$f \equiv f(\boldsymbol{\eta}(\mathbf{v}, \mathbf{x}, t), \tau(t)) \equiv F(\boldsymbol{\eta}, \tau) \tag{2.2}$$

$$\partial_\tau F(\boldsymbol{\eta}, \tau) = \mathrm{Col}\, F(\boldsymbol{\eta}, \tau) \tag{2.3}$$

Col F is the expression written down in (2.1) with $\mathbf{v}, \mathbf{w},...$ replaced by $\boldsymbol{\eta}_v$, $\boldsymbol{\eta}_w$,... always with $\boldsymbol{\eta}_v+\boldsymbol{\eta}_w=\boldsymbol{\eta}'_{v'}+\boldsymbol{\eta}'_{w'}$ and $\boldsymbol{\eta}_v^2+\boldsymbol{\eta}_w^2=\boldsymbol{\eta}'^2_{v'}+\boldsymbol{\eta}'^2_{w'}$.

Let us assume that $\boldsymbol{\eta}=\gamma(\mathbf{x}, t)[\mathbf{v}-\mathbf{v}_0(\mathbf{x}, t)]$, with γ and $\mathbf{v}_0$ unknown, this ansatz being suggested by the local Maxwellians of both (2.1) and (2.3). We want to show that γ and $\mathbf{v}_0$ can be expressed in terms of the temperature T and the mean velocity $\langle \mathbf{v} \rangle$ of the inhomogeneous solution. Furthermore, we establish the relations between the macroscopic quantities in both formalisms. For the homogeneous F we define constants of local mass, energy, momentum, and temperature

$$N_i = \int F |\boldsymbol{\eta}|^i \, d\boldsymbol{\eta}, \quad i=0, 2 \qquad N_0 \langle \boldsymbol{\eta} \rangle = \int F\boldsymbol{\eta} \, d\boldsymbol{\eta} \tag{2.4}$$

We choose the temperature of the homogeneous solution to be equal to 1; for the constants N_i we have $N_2/N_0=d$ [we recall that $F \to N_0(2\pi)^{-d/2}\exp(-\boldsymbol{\eta}^2/2)$ when $\tau \to \infty$] and performing a Galilei transformation, we choose $\langle \boldsymbol{\eta} \rangle = 0$. For the inhomogeneous f we define ρ, $\langle \mathbf{v} \rangle$, and T:

$$\rho = \int f \, d\mathbf{v}, \qquad \rho \langle \mathbf{v} \rangle = \int f\mathbf{v} \, d\mathbf{v}, \qquad T\rho d = \int f\mathbf{c}^2 \, d\mathbf{v}, \qquad \mathbf{c} = \mathbf{v} - \langle \mathbf{v} \rangle \tag{2.5}$$

Substituting (2.2) and $\boldsymbol{\eta}=\gamma(\mathbf{v}-\mathbf{v}_0)$ into (2.5), taking into account (2.4) and $d\boldsymbol{\eta}=\gamma^d\, d\mathbf{v}$, we find the relations between the inhomogeneous and homogeneous macroscopic quantities:

$$\rho = \gamma^{-d} N_0, \quad \rho\langle \mathbf{v} \rangle = \mathbf{v}_0 \gamma^{-d} N_0 = \rho \mathbf{v}_0, \quad dT\rho = N_0 \gamma^{-d} N_2 N_0^{-1} \gamma^{-2} = d\rho\gamma^{-2} \tag{2.6}$$

It follows that $\gamma^{-2} = T$, $\mathbf{v}_0 = \langle \mathbf{v} \rangle$, and

$$\boldsymbol{\eta} = T^{-1/2}(\mathbf{v} - \langle \mathbf{v} \rangle) = T^{-1/2}\mathbf{c}, \qquad \rho T^{-d/2} = \text{const} = N_0 \tag{2.7}$$

We rewrite the B.E. (2.1) for the inhomogeneous $f \equiv F$ with $\boldsymbol{\eta}$ written down in (2.7) and factorize out in Col f the factor not present in Col F (we perform the change of variable $\mathbf{v} \to \boldsymbol{\eta} T^{1/2} + \langle \mathbf{v} \rangle$)

$$\frac{d\tau}{dt} f_\tau + \sum_j f_{\eta_j} \mathscr{L}(\eta_j) = \mu T^{1 + [(d-1)/2](p-3)/(p-1)} \operatorname{Col} F \tag{2.8}$$

where η_j is the jth component of $\boldsymbol{\eta}$. If both conditions

$$\frac{d\tau}{dt} = \mu T^{1 + (d - 1/2)(p-3)/(p-1)} \tag{2.9}$$

$$\mathscr{L}(\eta_j) = 0 \qquad \forall j \tag{2.10}$$

are satisfied, then $f \equiv F$ satisfies the homogeneous B.E. (2.3). As we shall see later, $\mathscr{L}(\eta_j) = 0$ determines classes of compatible forces $\mathbf{A}$ and temperatures T (or density ρ with $\rho T^{-d/2} = \text{const}$). If we look at the large-time local Maxwellians, we find different possibilities:

1. If $T(t) \to 0$, then $\rho \to 0$, $f_{\mathrm{LM}} \to 0$, and the gas is in expansion. This is what happens if the forces are not present, as in the original Nikolskii transform.
2. If $T(t) \to \text{const}$, then $\rho \to \text{const}$, $f_{\mathrm{LM}} \to f_{\mathrm{AM}}$, an absolute Maxwellian.
3. If $T(t)$ oscillates, then ρ and f_{LM} also oscillate.

In the Nikolskii framework where $T \to 0$, a distinction occurs depending on whether the time integral on the rhs of (2.9) leads to $\tau \to \infty$ or a constant [equivalently, depending on whether the rhs of (2.9) decreases less or more than t^{-1}]. Because, as we shall see, $T \approx t^{-2}$, the two different behaviors appear when the parameter p of the intermolecular potential crosses the value $3 - 2/d$ ($7/3$ for $d = 3$). However, in both cases $T \to 0$ and the gas is in expansion. Here we introduce confining forces to keep temperatures from going to zero (whether the decrease of T is slow or fast). Consequently, we do not consider these distinctions.

The important point is that $\mathscr{L}(\eta_j) = 0$ alone determines T. Thus, the existence or not of absolute and oscillating Maxwellian relaxations becomes independent of the intermolecular forces. In other words, when the external forces are confining the gas, this property is independent of the microscopic interactions between the molcules. For instance, for hard spheres we can everywhere go to the limit $p \to \infty$ in the above equations without altering the confinement property. [On the rhs of (2.1) we have g^1, and the change of variable $\boldsymbol{\eta}$ instead of $\mathbf{v}$ leads to $T^{(1+d)/2}$ on the rhs of (2.8) and $d\tau/dt = \mu T^{(1+d)/2}$ in (2.9)].

If the homogeneous distributions E depend only on the energy, $F(\boldsymbol{\eta}^2, \tau)$, then the lhs of (2.8) becomes $(d\tau/dt) f_\tau + f_{\eta^2} \mathscr{L}(\boldsymbol{\eta}^2)$ and in (2.10) $\mathscr{L}(\eta_j) = 0$ is replaced by $\mathscr{L}(\boldsymbol{\eta}^2) = 0$. We notice that if $\mathscr{L}(\eta_j) = 0$, $\forall j$, then $\mathscr{L}(\boldsymbol{\eta}^2) = 0$, but the converse is not necessarily true.

2.2. $\mathscr{L}(\eta_j) = 0$ with Spatially Dependent Forces $\mathbf{A}(\mathbf{x}, t)$

We seek the compatible $\gamma = T^{-1/2}$, $\langle \mathbf{v} \rangle$, and $\mathbf{A}$ satisfying the linear differential equation

$$\left(\partial_t + \sum_i v_i \partial_{x_i} + A_i \partial_{v_i}\right) [\gamma(v_j - \langle \mathbf{v} \rangle_j)] = 0 \qquad \forall j \tag{2.11}$$

All the coefficients of the v_i powers must be zero. Then $v_i v_j$, v_j constant, gives, respectively,

$$\gamma = \gamma(t), \qquad \langle \mathbf{v} \rangle_{j,x_i} = \delta_{ij} \gamma_t \gamma^{-1}, \qquad A_j \gamma - \partial_t \gamma \langle \mathbf{v} \rangle_j = 0$$

from which we obtain the general solution

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &= a(t)\mathbf{x} + \mathbf{A}_0(t), \qquad a\gamma = \gamma_{tt} \\ \gamma \langle \mathbf{v} \rangle &= \gamma_t \mathbf{x} + \boldsymbol{\alpha}(0) + \int_0^t \mathbf{A}_0(') \, \gamma(t') \, dt' \end{aligned} \tag{2.12}$$

and the outside potential is a pure harmonic one.

1. If the force is independent of $\mathbf{x}$, then $a = 0$, $T = \gamma^{-2} = [c_0 + c_1 t]^{-2} \to 0$.

2. If $a(t) \not\equiv 0$, then the equation for $T^{-1/2}$ is like an S-wave Schrödinger equation for zero momentum. If the potential "$a(t)$" is "regular," both at $t = 0$ and $t = \infty$ or $t^2 a(t) \to 0$, then $T^{-1/2}$ is a "Jost solution,"

$$T^{-1/2} = \text{const} + \int_t^\infty (t' - t) \, T^{-1/2}(t') \, a(t') \, dt' \tag{2.13}$$

for which we obtain temperatures T going to a constant when $t \to \infty$. For instance, there exist explicit Bessel solutions for the temperature,

$$\begin{aligned} a(t) = a e^{-a_1 t} &\to T^{-1/2} = I_0(2\sqrt{a} \, e^{-a_1 t/2}) / I_0(2\sqrt{a}) \qquad &\text{if} \quad a > 0 \\ &\to T^{-1/2} = J_0(2\sqrt{-a} \, e^{-a_1 t/2}) / J_0(2\sqrt{-a}) \qquad &\text{if} \quad 0 < -a < 1.44 \end{aligned} \tag{2.14}$$

3. If $a(t)$ is oscillating, we can find temperatures T oscillating, too. We notice that we must require that $a(t)$ be such that $T > 0$.

2.3. $\mathscr{L}(\eta^2)=0$ with $\mathbf{A}(\mathbf{x}, t)$

The discussion is similar to the previous one, but more complicated, because the highest v_i power is of the third order. The analysis is made in Appendix A (see also Ref. 4), putting to zero the coefficients of the various powers $|\mathbf{v}|^2 v_i$, $v_i v_j$, v_i^2, v_i, const. We obtain a more general family of solutions than previously.

The general mean velocity is

$$\langle \mathbf{v} \rangle_i = \alpha_i(t) + \gamma^{-1}\gamma_t x_i + \sum_j \omega_{ij}(t) x_j \tag{2.15}$$

with ω_{ij} an antisymmetric tensor: $\omega_{ij}+\omega_{ji}=0$. Similarly, the foce $\mathbf{A}$, still linear in the spatial components x_i, can contain an antisymmetric tensor part and can furthermore be conservative or not, i.e.,

$$\partial_{x_j} A_i - \partial_{x_i} A_j = 2\gamma^{-2}\frac{d}{dt}\gamma^2\omega_{ij} \tag{2.16}$$

depending upon whether $\gamma^2\omega_{ij}$ is time independent or not. We distinguish the following possibilities:

1. $\mathbf{A}=0$ or the force is not present (see Appendix A1.2). For $\langle \mathbf{v} \rangle$ we obtain solutions purely proportional to $\mathbf{x}$ (or $\omega_{ij}=0$) as well as solutions with tensor term $\omega_{ij}\neq 0$ (unfortunately not for the physical $d=3$ case, but for $d=2, 4,\ldots$). For the temperature we find that γ^2 or T^{-1} is quadratic in the time variable and so $T\to 0$ when $t\to\infty$.

2. $\mathbf{A}(\mathbf{x}, t)$ is a conservative foce without antisymmetric part ($\omega_{ij}=0$). The solution is the same as the one for $\mathscr{L}(\eta_j)=0$ written down in (2.12). The external potential is purely harmonic and the temperature satisfies

$$\partial_{t^2}^2 T^{-1/2} = a(t)\, T^{-1/2}$$

with the same possibility of T going to a constant or oscillating.

3. $\mathbf{A}(\mathbf{x}, t)$ contains a part with the tensor ω_{ij} and is either conservative or not (see Appendix A1 and A1.3). The rhs of Eq. (A30) defines the different spatial dependences of the force:

$$A_i = \left[\gamma^{-1}\partial_t(\alpha_i\gamma) + \sum_j \alpha_j\omega_{ij}\right] + x_i\left(\gamma^{-1}\gamma_{tt} - \sum_j \omega_{ij}^2\right) + \sum_j x_j\left(\gamma^{-1}\partial_t\gamma^2\omega_{ij} + \sum_{\substack{l\\ i\neq j}} \omega_{il}\omega_{lj}\right) \tag{2.17}$$

If we call $a(t)\mathbf{x}$ the part of $\mathbf{A}$ proportional to $\mathbf{x}$, then, as above, we find $\gamma_{tt} = a\gamma$ with the possibilities of $T \to$ const or T oscillating. We can consider the $\omega_{ij}(t)$ as arbitrary functions and the last term defines the x_j-dependent part of the ith component of the foce $\mathbf{A}$. Similarly, we can consider the α_i terms of $\langle \mathbf{v} \rangle_i$ as arbitrary (we can also choose $\alpha_i = 0$) and the first bracket defines the part of the force that is independent of the space variables.

2.4. Connection between the Solutions of $\mathscr{L}(\eta^2) = 0$ and Those of $\mathscr{L}(f_{\mathrm{LM}}) = 0$ (see Appendix A)

There exists an interpretation of the conditions $\mathscr{L}(\boldsymbol{\eta}^2) = 0$ necessary for obtaining inhomogeneous similarity solutions associated with energy-dependent homogeneous ones. A long time ago Boltzmann[12] (see also Cercignani[13]) studied the time- and space-dependent local Maxwellians corresponding to a vanishing collision term. Let us rewrite $f_{\mathrm{LM}} = \nu(\mathbf{x}, t) \exp(-\boldsymbol{\eta}^2/2)$ with always $\boldsymbol{\eta}^2 = \mathbf{c}^2 T^{-1}$ and define $\nu = \rho(2\pi T)^{-d/2}$. If $\mathrm{Col}\, f_{\mathrm{LM}} \equiv 0$, we find

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) \log \nu(\mathbf{x}, t) \equiv \mathscr{L}(\boldsymbol{\eta}^2/2) \qquad \text{or} \qquad \mathscr{L}(f_{\mathrm{LM}}) \equiv 0 \tag{2.18}$$

In Appendix A we seek the sufficient assumption on ν such that (2.18) leads to $\mathscr{L}(\boldsymbol{\eta}^2) = 0$. From (2.18), assuming (see Appendix A1.3) ν (or $\rho T^{-d/2}$) independent of $\mathbf{x}$, then one proves that $\nu =$ const, whence $\mathscr{L}(\boldsymbol{\eta}^2) = 0$. We recall that this is the condition for the Nicholskii transform associated with $F(\boldsymbol{\eta}^2, \tau)$. It is shown in Appendix A1.1 that if $\mathscr{L}(f_{\mathrm{LM}}) = 0$, without force $\mathbf{A} = 0$, then necessarily $T, f_{\mathrm{LM}} \to 0$; if, further, ν is independent of $\mathbf{x}$, we discuss in Appendix A1.2 the two possibilities $\omega_{ij} = 0$ and $\omega_{ij} \neq 0$.

2.5. Maxwell Particles

The intermolecular potentials are such that g disappears in the collision term or $p - 1 - 2(d-1) = 0$. Then (2.8) can be rewritten

$$\tau - \tau(0) = \mu \int_0^t T^{d/2}\, dt' = \text{const} \times \int_0^t \rho(t')\, dt' \tag{2.19}$$

If the local density ρ of the inhomogeneous distribution decreases more slowly than t^{-1}, if $\rho \to$ const ($T \to$ const), or if ρ oscillates between two positive constant values, then $\tau \to \infty$ when $t \to \infty$. The time variable of the associated homogeneous distribution can go to infinity. In particular, when $\tau \to \infty$ ($t \to \infty$),

$$F \to (2\pi)^{-d/2} \exp(-\boldsymbol{\eta}^2/2) = (2\pi)^{-d/2} \rho(\infty)[2\pi T(\infty)]^{-d/2} \exp[-\mathbf{c}^2/2T(\infty)]$$

and $f \equiv F$ tends to an absolute Maxwellian if $T \to$ const. These remarks are not restricted to the Maxwell interaction, because for other intermolecular forces, if $T \to$ const, from (2.9), we see that $\tau \to \infty$ and $F \approx \exp(-\text{const} \times \mathbf{c}^2)$. However, for Maxwell particles and $F(\boldsymbol{\eta}^2, \tau)$ we know well the construction of explicit distributions and we restrict ourselves to these cases in Sections 3 and 4.

2.6. Local Entropy

We define $\mathscr{I}_0 = -\int f \log f \, d\mathbf{v}$ and $\mathscr{I} = \int \mathbf{v} f \log f \, d\mathbf{v}$ and assume spatially dependent external forces. From the standard treatment of the collision term it follows that $\partial_t \mathscr{I}_0 + \partial_{\mathbf{X}} \cdot \mathscr{I} \geqslant 0$. Further, assuming even inhomogeneous similarity distributions $f(\mathbf{c}^2/T, \tau(t))$ and introducing $\mathbf{c} = \mathbf{v} - \langle \mathbf{v} \rangle$, we find $\mathscr{I}_0 = \mathscr{I}_0(t)$ and $\mathscr{I} = \langle \mathbf{v} \rangle \mathscr{I}_0(t)$ (spatial dependence only through $\langle \mathbf{v} \rangle$). We recall from (2.15) and $\partial_{\mathbf{X}} \cdot \langle \mathbf{v} \rangle = -\frac{1}{2} \partial_t \log T$, whence $\partial_t \mathscr{I} T^{-d/2} \geqslant 0$. Finally we obtained that the local entropy density $\mathscr{I}_0$ divided by $T^{d/2}$ cannot be a decreasing function of the time.

3. HOMOGENEOUS FORMALISM REVISITED

We assume Maxwell intermolecular forces and again investigate the properties of $F(\boldsymbol{\eta}^2, \tau)$, a spatially homogeneous, energy-dependent distribution satisfying $\partial_\tau F = \mathrm{Col}\, F$. The connection with the inhomogeneous similarity distributions $f(\boldsymbol{\eta}^2 = \mathbf{C}^2 T^{-1}, \tau = \mu \int_0^t T^{d/2} \, dt')$ is obtained through the temperature T with $(\partial_{t^2}^2 - a(t)) T^{-1/2} = 0$ [$a(t)\mathbf{x}$ being the part of the external force proportional to $\mathbf{x}$). Mainly three methods have been developed for the construction of $F(\boldsymbol{\eta}^2, \tau)$.

1. The direct substitution of an appropriate ansatz into the homogeneous B.E. and the determination of the possible solutions. Only the BKW solution has been successfully built up in this way.[5]

2. The general theory following Grad's ideas uses expansion with Laguerre polynomials.[6,7] The Laguerre moments are recursively found from initial conditions at $\tau = 0$. Sufficient conditions in L^2 such that Laguerre moments remain in this space have been established.[14,2]

3. The last method used a generalized Laplace transform, which relates the distribution $F(\boldsymbol{\eta}^2, \tau)$ and the generating functional of the power moments. We must assume that the cross section has a well-defined behavior. The B.E. is replaced by a NL PDE[5,8,9] for which "solitons" and "bisolitons" have been determined[15] and in fact turn out to correspond to the BKW distribution. An elliptic Weierstrassian distribution was presented by Ernst[7,16] as a new exact solution. Independently, other authors[14]

were suspicious about this solution because numerical calculations seemed to indicate that it violates positivity. A rigorous proof was missing. Here, after an analytical study, we conclude that it must be rejected. A generalization of the NL PDE into an integrodifferential PDE was found by Ernst and Hendriks.[10] Here for this equation we determine the "solitons" and "bisolitons" and find again again that they correspond to the BKW solution.

3.1. Direct Use of the B.E.

Start with an ansatz that the solution is a product of a τ-dependent Gaussian $\exp[-\eta^2 b(\tau)]$ and an even polynomial $\sum^N a_n(\tau)\boldsymbol{\eta}^{2n}$. It turns out that the polynomial is necessarily quadratic, which leads to the BKW distribution

$$F(\boldsymbol{\eta}^2, \tau) = \frac{\exp[-\boldsymbol{\eta}^2/2K(1-\phi)]}{[2\pi K(1-\phi)]^{d/2}} N_0 \left[1 + \frac{\phi}{2(1-\phi)}\left(\frac{\boldsymbol{\eta}^2}{K(1-\phi)} - d\right)\right]$$

$$0 < \varphi(0) < (1 + d/2)^{-1} \tag{3.1}$$

$$\sigma_2 = \int \sin^2 \chi\, \sigma(\chi)\, d\Omega_d/4S_d$$

$$\phi = \phi(0) \exp(-\sigma_2 \tau), \qquad K = N_2(N_0 d)^{-1}$$

We can choose $K = 1$ if the constant homogeneous local density and energy satisfy $(N_2\, dN_0)^{-1} = 1$. The connection with the inhomogeneous distribution f was fully developed in Ref. 4. If the force $a(t)\mathbf{x}$ is such that T or ρ go to constants when $t \to \infty$, then $\tau \to \infty$, $\phi \to 0$, $F \to \text{const} \times \exp(-\eta^2/2)$, $f \to \text{const} \times \exp(-\mathbf{c}^2/2)$, an absolute Maxwellian. Similarity, if T or ρ oscillates, then $\tau \to \infty$, $\phi \to 0$, and asymptotically f becmes an oscillating Maxwellian.

3.2. Expansion with Laguerre Polynomials

We start with the Laguerre expansion

$$F = (2\pi)^{-d/2} \exp(-\boldsymbol{\eta}^2/2) \sum_0^\infty (-1)^n\, a_n(\tau)\, L_n^{d/2-1}(\boldsymbol{\eta}^2/2) \tag{3.2}$$

which we substitute into the homogeneous B.E. $\partial_\tau F = \text{Col}\, F$. The Laguerre moments $a_n(\tau)$ are solutions of a nonlinear differential system which can be integrated and included the i.c.

$$
\begin{aligned}
a_n(\tau) &= e^{-\beta_n \tau}\left[a_n(0) + \int_0^\tau \sum_1^{n-1} a_m(\tau')\, a_{n-m}(\tau') e^{-\beta_n \tau'} B_{m,n} C_n^m \, dt'\right] \\
B_{m,n} &= \frac{1}{2}\int_0^\pi d\chi\, \sigma(\chi) \operatorname{sn} \chi(\cos \chi^{2m}/2 + \sin \chi^{2(n-m)}/2 - \delta_{m,0}) \\
\sigma(\chi) &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)} \sigma^{(d)}(\chi) \sin \chi^{d-3} \\
\beta_0 = \beta_1 = 0, \qquad \beta_n &= \frac{1}{2} a_0 \int_0^\pi d\chi\, \sigma(\chi) \sin \chi(1 - \sin \chi^{2n}/2 - \cos \chi^{2n}/2)
\end{aligned} \tag{3.3}
$$

where we have introduced a pseudo cross section $\sigma(\chi)$, which reduces to $\sigma^{(3)}$ for $d=3$. As is usually done (although this is not necessary), we normalize the conserved quantities (local mass and local energy) such that $a_0 = 1$, $a_1 = 0$. The two first nontrivial Laguerre moments are a_2 and a_3, which decrease like $\exp(-\beta_2 \tau)$ and $\exp(-\beta_3 \tau)$, while all other moments have an increasing number of time-dependent terms, which are recursively determined from (3.3). For the existence of the solutions in L^2 spaces we introduce the norm

$$N(\tau) = \sum a_n^2(\tau) \frac{\Gamma(n + d/2)}{\Gamma(n+1)}$$

Sufficient conditions[9,14] on $N(0)$ such that $N(\tau) < \infty$ $\forall \tau \in [0, \infty]$ were established. Different assumptions occur, depending on the model of the cross section $\sigma(\chi)$. In Section 4, the integral recurrent relation (3.3) as well as the homogeneous time dependence$\tau(t)$ will be used for the explicit numerical construction of particular inhomogeneous similarity solutions $f(\boldsymbol{\eta}^2 = \mathbf{c}^2/T(t), \tau(t))$.

3.3. Nonlinear Partial Differential Equations Equivalent to Models of the Homogeneous Boltzmann Equation

We assume well-defined behavior for the cross section: $\sigma(\chi) \simeq (\sin \chi)^{q-1}$. Then the generating functional of the power moments, which is equivalent to a generalized Laplace transform of the distribution, satisfies a nonlinear partial differential equation, or more generally a nonlinear integrodifferential equation. In this subsection $F(u, \tau)$ is the homogeneous, energy (u)-dependent distribution, with Maxwellian $e^{-u/(1-v)}(1-v)^{-d/2}$ (v being a constant). Local density and energy

$$N_i = [\Gamma(d/2)]^{-1} \int_0^\infty u^{d/2-1} u^{i/2} F(u, \tau)\, du \qquad i = 0, 2$$

give for these conserved quantities 1 and $(1-v)d/2$, respectively. Let us define the transform

$$H(p, \tau)=\frac{(1+p)^q}{\Gamma(d/2)}\int_0^\infty du\, u^{d/2-1}\, {}_1F_1\left(q, \frac{d}{2}, -pu\right) F(u, \tau) \tag{3.4}$$

with $\sigma(\chi)=(\sin^2\chi)^{q-1}$, identical to $\sigma^{(d=3)}(\chi)$, while other d-dimensional cross-sections are deduced from (3.3).

We start with $q=1$ or an isotropic $d=3$ cross section. Equation (3.4) leads to the most popular NL PDE associated with the B.E. It was first obtained by Krook and Wu[5] for $d=3$, by Tjon and Wu[6] for $d=2$, and generalized for any d value,[9]

$$(\partial_\tau+1)(\partial_x+1)\, H\left(x=\log\frac{p}{p+1}, \tau\right)=H^2 \tag{3.5}$$

Some years ago this equation started the revival of interest in the Boltzmann equations. Krook and Wu got from it their BKW distribution. While from Grad's work the Laplace expansion was already understood in the 1960s, Eq. (3.5) made it possible to study the B.E. with the modern view of NL PDE. All the kinematic difficulties of the B.E. disappear in (3.5). Although (3.5) is not completely integrable, "solitons" and "bisoliton" solutions can be considered if we retain the mathematical generalization of rational fractions with one or two exponentials $(\exp(\gamma_i x+\rho_i t)$. Unfortunately, all the solutions of (3.5) cannot be retained as physical distributions of the B.E. On the one hand, the transform $H(p, \tau)$ must be such that F satisfies positivity; on the other hand, local density and energy must be constants independent of time. A look at the transform (3.4) taking into account ${}_1F_1 \simeq 1-2puq/d$ shows that $H(p=0, \tau)=N_0$ and $-d/2\, q\partial_p[H/(1+p)^q]_{p=0}=N_2$ must be independent of time.

For other integer values of $q \neq 1$ (or other cross sections) there exists a class of NL PDE that generalize (3.5),

$$(\partial_\tau+1)\prod_0^{q-1}\left(1+\frac{\partial_x}{q+m}\right) H\left(x=\log\frac{p}{p+1}, \tau\right)=H^2 \tag{3.6}$$

and reduce to (3.5) for $q=1$. The only known closed solutions[15] (not rejected by the physical constraints of the B.E.) are "solitons" and "bisolitons,"

$$H(p, \tau)=\frac{1-\omega_2-(q+1)\omega_1}{(1+\omega_1+\omega_2)^{q+1}} \tag{3.7}$$

with $\omega_2 = \nu_2 p/(p+1)$, $\omega_1 = \nu_1 p(p+1)e^{\rho\tau}$, $\rho = -q/2(2q+1)$. Here ν_1 and ν_2 are constants and $\nu_2 = 0$ or $\omega_2 = 0$ leads to the "solitons."

The general transform (3.4) was proposed by Ernst and Hendriks.[10] They deduced for H in the general q case an integrodifferential equation. The NL PDE (3.6) are particular equations of their integral equation when q becomes an integer. The Ernst–Hendriks integrodifferential equation is studied in Appendix B. We show that the "solitons" and "bisolitons" (3.7) still exist for q noninteger, the only difference being that the denominator does not have an integer power and we must still extend the usual definition of solitons and bisolitons. They lead to positive and physically relevant Boltzmann distributions, which again correspond to the BKW solutions.

In order to come back to the Boltzmann distribution we must invert the transform (3.4). It is convenient at an intermediate stage to write F as a Laguerre series and H as a $p^n/(1+p)^{n+q}$ series (see Appendix B.3) and then sum up the Laguerre series. The "bisolitons" (3.7) give:

$$F(u,\tau) = \frac{e^{-u/(1-z)}}{(1-z)^{d/2}}\left[1 + \frac{z-\nu_2}{1-z}\left(\frac{u}{1-z} - \frac{d}{2}\right)\right], \quad z = \nu_1 e^{\rho\tau} + \nu_2, \quad \rho = \frac{-q}{2(2q+1)} \tag{3.8}$$

The positivity requires $\nu_1 > 0$, and either $\nu_1 + \nu_2 < 1$, $\nu_1 < (1-\nu_2)(1+d/2)^{-1}$ or $\nu_1 + \nu_2 < 0$. The Maxwellian equilibrium state is $(\exp - u/(1-\nu_2))(1-\nu_2)^{-d/2}$ and $\nu_2 < 1$. For the conservation of mass and energy we note that $N_0 = 1$, $N_2 = d(1-\nu_2)/2$ are constants independent of time. When $\nu_2 = 0$, (3.8) is associated with the "solitons" (3.7) and (3.8) becomes the standard BKW solution. When $\nu_2 \neq 0$, (3.8) is in fact still a BKW solution obtained in subsection (3.1) by the direct use of the Boltzmann equation. In fact the only change is in the normalization of the Maxwellian relaxation.

It is unfortunate that up to now, if we except the BKW distributions no other exact solutions are known. It is usual to define the homogeneous similarity solutions as those for which the Laguerre moments factorize out only one time dependence. They have a group theoretical interpretation as was shown by Bobylev.[2] For the $q = 1$ case (isotropic $d = 3$ $\sigma(\chi)$) it was shown for the Laguerre series[14] (with normalization $a_0 = 1$, $a_1 = 0$) that outside this class the number of time dependences for the nth Laguerre moment increases with n with little hope of finding a closed solution. In terms of transform $H(p,t)$ the similarity solutions correspond to solutions with only one variable $p/p+1$ multiplied by some time dependence. The "solitons" are of this kind but not the "bisolitons." The existence of bisolitons for $q = 1$ (isotropic $d = 3$ $\sigma(\chi)$) does not contradict our previous analysis[14] because the normalization of the Laguerre moments $a_n \approx z^n\{1 - n[(z - v_2)/z]\}$ are different, for instance $a_1 \approx v_2$ is different from zero for $v_2 \neq 0$.

However, in the literature for $q = 1$, another exact homogeneous similarity solution was proposed by Ernst[16,7]—a Weierstrassian elliptic solution. Here we show analytically that this elliptic solution necessarily violates positivity and must be rejected. The study is done in Appendix B.4. The solution is an anharmonic Weierstrass elliptic function[17,16,14,7]

$$H(p, \tau) = 1 - \omega^2 \mathscr{P}(y = \omega + \alpha,\ g_2 = 0,\ g_3),\ \omega = e^{-\tau/6} p/(p+1) \tag{3.9}$$

α being an arbitrary constant. If $g_3 = 0$, $\alpha = 1$, then the degenerate elliptic solution gives back the BKW soliton (3.7) with one pole in the y complex plane. For $g_3 \neq 0$, with an infinite number of poles for $\mathscr{P}$, it is sufficient to study the doubly periodic function in a fundamental period parallelogram. For $p > 0$ (or $\omega > 0$) and g_3 real, it is shown in Appendix B that H tends to negative infinite values an infinite number of times; while H violates positivity. A look at the transform (3.4) $F \to H$ where the kernel ${}_1F_1(1, d/2, -pu)$ is positive (for $p > 0$, $d \geqslant 2$) shows that necessarily H should be positive if F was positive.

4. NUMERICAL CALCULATIONS, TJON EFFECT FOR THE ABSOLUTE MAXWELLIAN RELAXATION, OSCILATING MAXWELLIANS

For the numerical constructions of the inhomogeneous similarity distributions we must specify both the microscopic model of intermolecular forces, the initial conditions (at $t = 0$ or $\tau = 0$ both homogeneous and inhomogeneous distributions coincide) and the outside forces.

We choose the true Maxwellian interaction ($p = 5$ for $d = 3$ in (2.1)). Then the cross section $\sigma(\kappa)$ is singular like $\kappa^{-5/2}$ near $\kappa = 0$. The moments $B_{m,n}$ (see Eq. (3.3)) of the cross section are not independent. We define as in Ref. 18 a set of moments ϕ_n of $\sigma(\chi)$ and the $B_{m,n}$ are expressed as linear combination of these ϕ_n. The first moment ϕ_1 # 3.69 and the first ten ϕ_n were computed in Ref. 18. For the singular cross section $\sigma(\chi)$, the ϕ_n are given by

$$2^{2n}(\Gamma(n+1))^2 \phi_n = \Gamma(2n+2) \int_0^1 du (1+u) u^{-3/2} \sin^{2n} \chi(u)$$

$$\chi(u) = \pi - (1-u)^{1/2} \int_0^{\pi} d\theta (1 + u \cos^2 \theta)^{-1/2} \tag{4.1}$$

From the initial distribution we deduce the Laguerre moments $a_n(0)$ at $\tau = 0$, construct numerically the $a_n(\tau, t)$ from the Laguerre integral system (3.3) and build up the distributions from the Laguerre expansion $L_n^{d/2-1}(\mathbf{c}^2/2T(t))$. We find $a_0 = 1$, $a_1 = 0$, $a_2 = a_2(0) \exp - \phi_1 \tau(t)/3$, $a_3 = a_3(0) \exp - \phi_1 \tau(t)/2$ and the higher moments are recursively deduced. We have numerically constructed the first twelve $a_n(\tau(t))$. For the i.c. we choose simple families of distributions with known Laguerre expansions[14-18]

$$1 - z_1 z_2 \sum L_n^{(\alpha)}(x) \sum_{m+m'=n} z_1^m z_2^{m'}$$

$$= (z_1 - z_2)^{-1} \left(\frac{z_2}{(1-z_1)^{\alpha+1}} z^{xz_1/z_1 - 1} - \frac{z_1}{(1-z_2)^{\alpha+1}} e^{xz_2/z_2 - 1} \right) \quad (4.2)$$

$$\sum L_n^{(\alpha)}(x) z^n \sum_{p=0}^{q} d_p C_n^p$$

$$= (1-z)^{-(1+\alpha)} \exp\left(\frac{xz}{z-1}\right) \sum_{p=0}^{q} dp \left(\frac{z}{1-z}\right)^p L_p^{(\alpha)}\left(\frac{x}{1-z}\right) \quad (4.3)$$

with $\alpha = d/2 - 1$. At $\tau = 0$, we can numerically check the truncation of the Laguerre series. For the absolute Maxwellian relaxation $T(\infty) = \text{const}$, we choose the force $\mathbf{x}a \exp(-a_1 t)$ and the Bessel solution J_0 (if $a < 0$) and I_0 (if $a > 0$) written down in (2.14). For the oscillating Maxwellians, we assume oscillating T and deduce the force $a(t) = T^{1/2} \partial_{t^2}^2 T^{-1/2}$. In both cases the homogeneous time satisfies $\tau = \mu \int_0^t T^{d/2} \, dt$. All the numerical calculations have been done for the physical $d = 3$ case.

4.1. Absolute Maxwellians and Tjon Effect[11]

When this effect occurs, there exists an overpopulation of high-velocity particles at intermediate times. When $t \to \infty$, $f \simeq \text{const} \times \exp[-\mathbf{c}^2/2T(\infty)]$, let us define the reduced distribution

$$F_R = \frac{f(\mathbf{v}, \mathbf{x}, t)}{f(\mathbf{v}, \mathbf{x}, \infty)}$$

$$= \left(\exp \delta T \frac{\mathbf{c}^2}{2T} \right) \left[1 + \sum a_n(\tau(t))(-1)^n L_n^{d/2-1} \frac{\mathbf{c}^2}{2T} \right] = F_R(|\mathbf{c}|, t) \quad (4.4)$$

with $\delta T + 1 = T(t)/T(\infty)$. Depending upon whether the effects exists or not, the relaxation toward 1 is from above, $F \to 1^+$, or from below, $F \to 1^-$. (However, the effect really exists if the F values are substantially larger than 1 at intermediate time.) In the homogeneous formalism T is a constant, the first factor in (4.4) is 1, and $F \to 1^+$ if the first term of the expansion in (4.4) [which asymptotically is the dominant one, $\lim_{\tau \to \infty} a_n(\tau)/a_0(\tau) = \delta_{n0}$] is positive. This gives $a_2(0) > 0$ and constitutes the Hauge–Praestgaard criterion on the i.c. of the distribution. As a consequence, i.c. like (4.2) with only one bump and $a_2(0) < 0$ cannot lead to the effect.[18] In the inhomogeneous formalism $\delta T \not\equiv 0$, both factors in (4.4) contribute. Let us look at the Gaussian term $\exp[\delta T(\mathbf{c}^2/2T)] = G$ in (4.4), $G \to 1^+$ (or 1^-) depending upon whether T decreases (or increases) toward $T(\infty)$. From (2.13) we have $\partial_t T^{-1/2} = -\int_t^\infty T^{-1/2} a(t')\,dt'$, whence $\delta T > 0$ if $a < 0$ ($\delta T < 0$ if $a > 0$). In other words, the temperature T decreases (increases) if the external potential is attractive (repulsive). Consequently, different possibilities can occur in (4.4):

1. Repulsive external forces ($a > 0$, $G \to 1^-$) and i.c. as in (4.2) with $a_2(0) = -z_1 z_2 < 0$. We predict no effect, and this is well verified in Fig. 1. Inhomogeneous similarity solutions depend in fact on two variables, which we choose to be $|\mathbf{c}|$ and t. Of course $|\mathbf{c}| = |\mathbf{v} - \langle \mathbf{v}(\mathbf{x}, t) \rangle|$ depend on $\mathbf{v}$, $\mathbf{x}$, t here, $\langle \mathbf{v} \rangle = -\mathbf{x}/2\, \partial_t \log T(t)$, with T given in (2.14). The i.c. for $F_R(|\mathbf{c}|, t)$ means $F_R(|\mathbf{c}(t=0)|, 0)$, but in the study of the relaxation curves, we do not take into account the t dependence of $|\mathbf{c}|$ because we consider t and $|\mathbf{c}|$ as independent variables.

2. Attractive external forces ($a < 0$, $G \to 1^+$) and i.c. such that $a_2(0) > 0$ [examples of this type can be built up with (4.3)]. Then we predict the effect.

3. Mixed situations where both factors in (4.4) are going in opposite limits $1^\pm$ are more appealing. New features which disagree with the homogeneous predictions could occur. Among the two possibilities, we choose the one where the Hauge–Praestgaard criterion predicts no effect $a_2(0) < 0$, but is challenged by an attractive outside force (J_0 solution for $T^{-1/2}$). In Fig. 2 for the same i.c. (4.2) and the force $\mathbf{x}a \exp(-a_1 t)$ with the same a but different a_1 values (or different decreases of the force) we observe contradictory results, with the effect in one case and no effect in the other. So, another feature appears in the prediction of the effect. We expand both factors in (4.4), keeping the dominant terms

$$F_R - 1 \simeq \left(\frac{\mathbf{c}^2}{2T}\right) a_2 \left[-\left(\frac{d}{2} - 1\right) + \frac{\delta T}{a_2} + \left(\frac{\mathbf{c}^2}{4T}\right)\left(1 - \delta T(d+2) + \frac{(\delta T)^2}{a_2}\right)\right] \quad (4.5)$$

and look at $t \to \infty$ for $\mathbf{c}^2$ fixed. We must compare the decrease of the temperature, which is like $\delta T \simeq \exp(-a_1 t)$, and the decrease of the second

Laguerre moment $a_2(t)$, which is like $\exp[-\phi_1\tau(t)/3]$ or $\simeq\exp[-t(\phi_1/3 \lim \tau/t)]$, where $\lim \tau/t$ represents the asymptotic dilatation (or contraction) of the homogeneous time with respect to the inhomogeneous one; here $\lim \tau/t=\mu T(\infty)^{d/2}$. We define a criterion

$$\text{crit}=\frac{6a_1}{\phi_1}\frac{1}{\lim(\tau/t)}\rightarrow\begin{cases}\text{crit}\ll 1\rightarrow F\rightarrow 1^+\\ \text{crit}\gg 1\rightarrow F\rightarrow 1^-\end{cases}$$

In Fig. 2, $\phi_1/3\simeq 1$, $\lim \tau/t\simeq 10$, and $\text{crit}\simeq 0.16a_1$. In Fig. 2a, $a_1=1$, we observe the effect and in Fig. 2b, $a_1\simeq 10$, we see the relaxation from below. We do not push too far the validity of this rough estimate and only retain small and large values compared to crit. In fact, many terms have been neglected in (4.4); note also that the ratios $\delta T/a_2$ and $(\delta T)^2/a_2$ occur. On

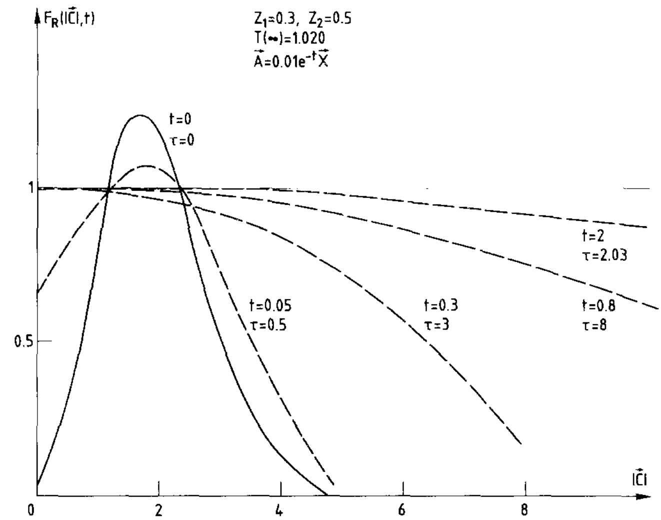
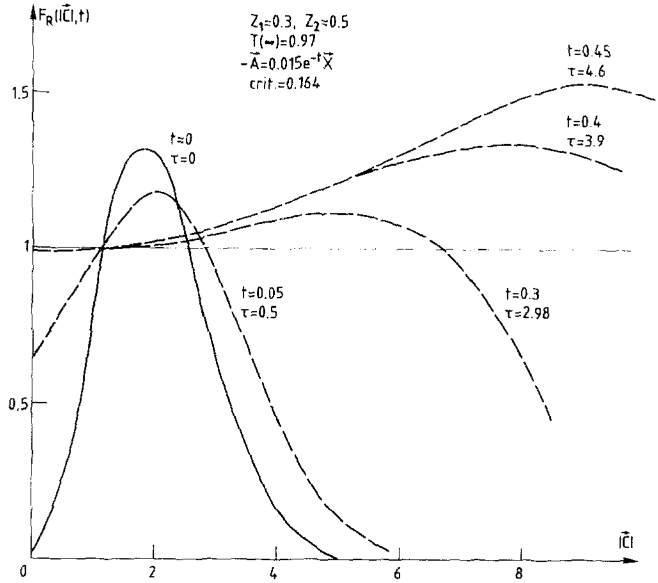


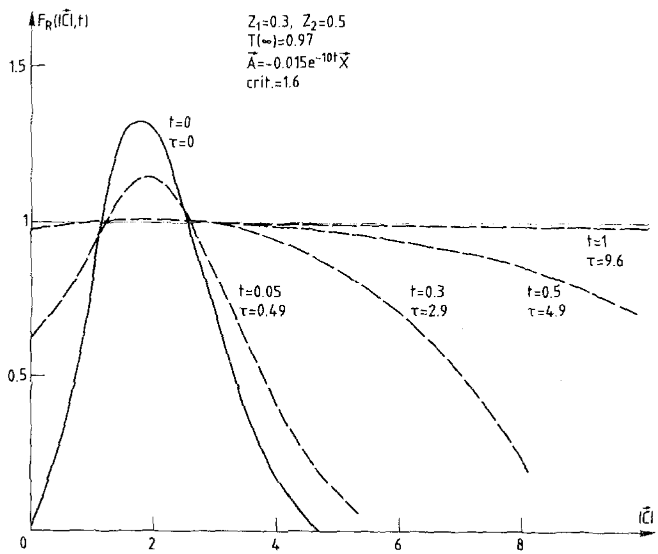
Fig. 1. Plot of the reduced $F_R(|\mathbf{c}|, t)$ against $|\mathbf{c}|$ for the external force $\mathbf{A}=ae^{-a_1t}\mathbf{x}$ and i.c. $F_R(|\mathbf{c}|, 0)=f(|\mathbf{c}|, 0)/f(|\mathbf{c}|, \infty)$, with $f(|\mathbf{c}|, \infty)=(2\pi)^{-3/2}\exp[-\mathbf{c}^2/2T(\infty)]$, $\mathbf{c}=\mathbf{v}+X/2\,\partial_t \log T(t)$,

$$F_R(|\mathbf{c}|, 0)=\exp\left[\mathbf{c}^2\frac{1}{2}\left(\frac{1}{T(\infty)}-\frac{1}{T(0)}\right)\right]\times\left[z_2(1-z_1)^{-3/2}\exp\left(\frac{-\mathbf{c}^2z_1}{2T(0)(1-z_1)}\right)+z_1\leftrightarrow z_2\right]$$

$T(0)=1$, $T(\infty)=[I_0(2a^{1/2})]^{-2}$, $T(t)/T(\infty)=[I_0(2a^{1/2}e^{-a_1t/2})]^{-2}$, $\tau(t)=\mu\int_0^t T^{3/2}\,dt'$; $a=0.01$, $z_1=0.3$, $z_2=0.5$, $T(\infty)=1.02$, $\mu=10$, $a_1=1$.



(a)



(b)

Fig. 2. As in Fig. 1, but for $a=0.015$, $T(\infty)=[J_0(2|a|^{1/2})]^{-2}=0.97$, $T(t)/T(\infty)=[J_0(2|a|^{1/2}e^{-a_1t/2})]^{-2}$; (a) $a_1=1$, crit $=0.16$; (b) $a_1=10$, crit $=1.6$.

the other hand, we know that the third Laguerre moment $a_3(0)$, not present in (4.4), is important for the effect.[4] Intuitively, the result is due to two competing opposite effects. We choose an i.c. that favor the relaxation from below, and this remains true if the outside foce decreases too fast (as in Fig. 2b), while if the force decreases slowly, then the relaxation from above will be dominant. The important point is that we must take into account the microscopic model of the cross section (here ϕ_1 indicates the decreases of a_2; notice that for the inhomogeneous similarity BKW exact solution, the same features were observed in Ref. 4). In conclusion, the effect depends not only on the i.c. (as in the homogeneous formalism), but also on the decrease of the external force, on the sign of the force (attractive or repulsive), and finally on the microscopic model of the interaction. The existence or not of the effect is a signature of all the properties of the Boltzmann model at hand.

4.2. Oscillating Maxwellians

We start with an oscillating temperature $T = [1 + r(\sin t + \lambda \sin qt)]^{-2}$ and choose q integer, with λ an arbitrary parameter and with an outside force that is also oscillating. The oscillating relaxations have been thoroughly studied in Ref. 4 for the exact BKW inhomogeneous similarity solutions. Here we want only to show that the results are more general. If we choose $q = 2$, then for $|\lambda| < 0.5$, when the distributions are in the asymptotic regime, they oscillate between two Maxwellians and four for $|\lambda| > 0.5$. In Figs. 3a and 3b for $\lambda = 0$ and $\lambda = 1$, with i.c. provided by (4.3), we observe these two modes of relaxation. Starting from $t = 0$, we observe two successive regimes: first a preasymptotic regime and second the asymptotic one where the distribution oscillates between different Maxwellians.

5. LINEAR VELOCITY-DEPENDENT FORCES AND UNIFORM SOURCES

Our aim is to extend all results of the previous sections to another type of external force: linear velocity-deoendent forces plus source terms. We still assume intermolecular forces with inverse power law, but the external forces are uniform in space. We start with the inhomogeneous distribution $\mathscr{L}f(\mathbf{v}, \mathbf{x}, t) = \mu \operatorname{Col} f$:

$$\mathscr{L} = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} + a_1(t) \partial_{\mathbf{v}} \cdot \mathbf{v} + a_2(t) + \mathbf{A}(t) \cdot \partial_{\mathbf{v}}, \qquad \bar{\mathscr{L}} = \mathscr{L} - a_1 d - a_2 \quad (5.1)$$

[4] In this Ref. 18 the first ten numerical ϕ_n values as well as the fist 12 differental equations for the Laguerre moments $a_n(t)$ ar quotd.

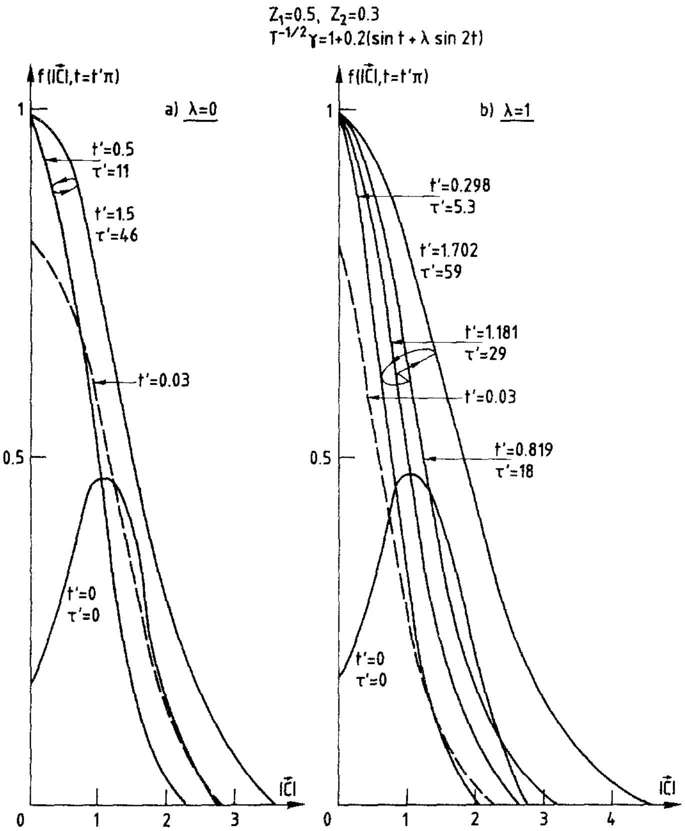


Fig. 3. Plot of $f(|\mathbf{c}|, t)$ against $|\mathbf{c}|$ for $T^{-1/2} = 1 + r(\sin t + \lambda \sin 2t)$ and i.c.

$$f(|\mathbf{c}|, 0) = \{\exp[-\mathbf{c}^2/2T(0)]\}(1-z_1)^{-3/2}[\exp(-\omega^2 z_1)]$$
$$\times [1 + yL_1^{3/2}(\omega^2) + z^2y^2L_2^{3/2}(\omega^2)]$$

$y = z_1(1-z_1)^{-1}$, $\omega^2 = \mathbf{c}^2[2T(0)(1-z_1)]^{-1}$, $T(0) = 1$, $t = t'\pi$, $\tau = \tau'\pi$, $\tau = \mu \int_0^t T^{-3/2}\, dt'$; $z_1 = 0.5$, $z_2 = 0.3$, $\mu = 10$, $r = 0.2$; (a) $\lambda = 0$, (b) $\lambda = 1$.

and define the differential part of $\mathscr{L}$:

$$\bar{\mathscr{L}} = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} + [a_1(t)\mathbf{v} + \mathbf{A}(t)] \cdot \partial_{\mathbf{v}}$$

with Col f still defined by (2.1′).

We will associate an homogeneous formalism with the source term. As we shall see, two cases occur, depending upon whether $a_1 d + a_2 = 0$ or not. In the first case the formalism is almost identical with the spatially dependent force, the main difference being the relation between the temperature T and the force $a_1(t)$.

5.1. Nikolskii Transform

We define the inhomogeneous similarity solution $f(\boldsymbol{\eta}(\mathbf{v}, \mathbf{x}, t), \tau(t))$ identical to an homogeneous distribution $F(\boldsymbol{\eta}, \tau)$, the only difference with Section 2 being that the homogeneous formalism contains a source term,

$$[\partial_\tau + a(\tau)]\, F(\boldsymbol{\eta}, \tau) = \mathrm{Col}\, F(\boldsymbol{\eta}, \tau) \tag{5.2}$$

We still assume $\boldsymbol{\eta} = \gamma(\mathbf{v} - \mathbf{v}_0)$, γ and $\mathbf{v}_0$ being unknown, and introduce homogeneous and inhomogeneous macroscopic quantities N_0, N_2, $\langle \boldsymbol{\eta} \rangle$ and ρ, T, $\langle \mathbf{v} \rangle$. The main difference is that the N_i are not constants if $a(\tau) \neq 0$. Let us define $b(\tau) = \exp - \int_0^\tau a(\tau')\, d\tau'$, and assume $N_2(0) = dN_0(0)$; then $N_i = N_i(0)\, b(\tau)$, $\langle \boldsymbol{\eta} \rangle = \langle \boldsymbol{\eta}(0) \rangle$, but we still choose $\langle \boldsymbol{\eta} \rangle = 0$, $N_2(0)/N_0(0) = d$ [Note that $F_{\mathrm{LM}} = b(\tau)(2\pi)^{-d/2} \exp(-\boldsymbol{\eta}^2/2)$.] All relations (2.6)–(2.7) are valid except $\rho T^{-d/2} = N_0(0)\, b(\tau)$, which explains the supplementary term in F_{LM}. Finally, we still have $\gamma = T^{-1/2}$, $\mathbf{v}_0 = \langle \mathbf{v} \rangle$, and $\boldsymbol{\eta}^2 = \mathbf{c}^2 T^{-1}$. Performing the change of variable $\mathbf{v} \to \boldsymbol{\eta}$ in Col f and factoring out the temperature (or γ)-dependent term, it follows, as in Section 2, that the inhomogeneous B.E. reduces to the homogeneous one (5.2), provided a supplementary relation $(d\tau/dt)\, a(\tau) = a_1 d + a_2$ is satisfied. We have

$$\begin{gathered} \frac{d\tau}{dt} = \mu T^{1 + [(d-1)/2](p-3)/(p-1)}, \qquad \mathscr{L}(\eta_j) = 0 \\ \int_0^t (a_1 + da_2)\, dt' = \int_0^\tau a(\tau')\, d\tau' \end{gathered} \tag{5.3}$$

As for the spatial external force, the condition $\mathscr{L}(\eta_j) = 0$ [$\mathscr{L}(\boldsymbol{\eta}^2) = 0$ if $F = \mathbf{F}(\boldsymbol{\eta}^2, \tau)$] represents the condition on the outside force which alone determines the temperature T. The possibility of connecting homogeneous and inhomogeneous formalism is still independent of the intermolcular forces (for instance, we can as well have hard spheres). Of course, the homogeneous time τ in (5.3) depends on the inverse power law p, but, for instance, if $T \to$ const, then in all cases $\tau \to \infty$.

5.2. $\mathscr{L}(\eta_j) = 0$

We remark that the source term $a_2(t)$ disappears in $\mathscr{L}$, so that only conditions on the external forces remain. We have

$$\left(\partial_t + \sum \{v_i \partial_{x_i} + [a_1(t) v_i + A_i(t)]\, \partial_{v_i}\} [\gamma(v_j - \langle \mathbf{v} \rangle_j)]\right) = 0 \tag{5.4}$$

As for the spatial forces, the coefficients of $v_i v_j$, v_i, v_j, const are zero. We find

$$\gamma_{x_i} = \langle \mathbf{v} \rangle_{j,x_i} = \partial_t(a_1 \gamma + \gamma_t) = 0, \qquad \langle \mathbf{v} \rangle_{j,x_j} = a_1 + \partial_t \log \gamma$$

and

$$\begin{gathered} \gamma \langle \mathbf{v} \rangle = \boldsymbol{\alpha}(0) + \int_0^t \mathbf{A}(t')\, \gamma(t')\, dt' + \nu_0 \mathbf{x}, \qquad a_1 = (\nu_0 - \gamma_t)\gamma^{-1} \\ \gamma = \left(\exp - \int_0^t a_1\, dt' \right)\left(1 + \nu_0 \int_0^t \exp \int_0^{t'} a_1\, dt''\, dt' \right) = T^{-1/2} \end{gathered} \tag{5.5}$$

ν_0 being a constant. If we except the relation between temperature and force, all other relations are similar to the spatial force ones. We can still have oscillating T or $T \to$ const. For instance, with the constant positive force $a_1 = a_1(0)$

$$\gamma = T^{-1/2} = \frac{\nu_0}{a_1(0)} (1 - e^{-a_1(0)t}) + e^{-a_1(0)t} \to \frac{\nu_0}{a_1(0)} \tag{5.6}$$

Of course other forces lead to $T(\infty) = \text{const}$; for instance, let us choose arbitrary $\varepsilon(t) \to 0$, $\varepsilon_t \to 0$, and $\gamma = \gamma(\infty) + \varepsilon(t)$. We deduce the force $a_1 = (\nu_0 - \varepsilon_t)/[\gamma(\infty) + \varepsilon(t)] \to \nu_0/\gamma(\infty)$. However, as we shall see, this is not sufficient to obtain absolute Maxwellians. The weaker condition $\bar{\mathscr{L}}(\boldsymbol{\eta}^2) = 0$ is studied in Appendix A (see Ref. 4. We can have mean velocity $\langle \mathbf{v} \rangle$ of type (5.5), but with a supplementary antisymmetric tensor part $\sum_j \omega_{ij} x_j$ (however, not for the physical $d = 3$ case).

5.3. Local Maxwellian and $\mathscr{L}(\boldsymbol{\eta}^2) = 0$

The study of $\bar{\mathscr{L}}(\boldsymbol{\eta}^2) = 0$ alone as well as the connections between the conditions $\mathscr{L}(\nu(\mathbf{x}, t) \exp(-\boldsymbol{\eta}^2/2)) = 0$ with either $\bar{\mathscr{L}}(\eta_j) = 0\ \forall j$ or $\bar{\mathscr{L}}(\boldsymbol{\eta}^2) = 0$ is done in Appendix A. If we compare with the spatial force, a new pure time factor appears,

$$f_{\mathrm{LM}} = \frac{1}{(2\pi)^{d/2}} \exp - \int_0^t (a_1 d + a_2)\, dt' \exp - \frac{\mathbf{c}^2}{2T} \tag{5.7}$$

with T provided by the force as in (5.5).

If $a_1 d + a_2 = 0$, only the usual Gaussian term remains and we can, for instance, have absolute Maxwellians $T \to T(\infty)$ and $f_{\mathrm{LM}} \to \text{const} \times \exp[-\mathbf{c}^2/2T(\infty)]$ as in the spatial case. If $\mathscr{L} f_{\mathrm{LM}} = 0$, $a_1 d + a_2 = 0$, $\nu = \nu(t)$, it is shown in Appendix A2.1 that necessarily $\nu = \text{const}$ and $\mathscr{L}(\boldsymbol{\eta}^2) = 0$. If

$a_1 d + a_2 \neq 0$, then, as was shown in Ref. 4, both Gaussian and pure time factor terms lead to incompatible asymptotic behaviors (with the meaning that for $|\mathbf{c}|$ fixed, $t \to \infty$, the product of the two terms is singular, going either to zero or to infinity). However, if the force is such that $T \to 0$, then a meaning can be given to the asymptotic behaviors. For instance, for the pure attractice force $a_1(t) = -a_1(0)$,

$$T \to [\exp - 2a_1(0)t] \left[1 + \nu_0 \int_0^t \exp - a_1(0)t' \, dt' \right]^{-2} \to 0$$

and

$$f_{\mathrm{LM}} \to \mathrm{const} \times \delta(\mathbf{v} - \langle \mathbf{v}(t = \infty) \rangle)$$

5.4. Maxwell Particles and $a_1 d + a_2 = 0$

For this particular mixing, the external force and source term can be rewritten $[a(t)\mathbf{v} + \mathbf{A}(t)] \cdot \partial_{\mathbf{V}}$. In this subsection we restrict our study to intermolecular forces $p - 1 = 2(d-1)$ and energy-dependent homogeneous solutions $F = F(\boldsymbol{\eta}^2, \tau)$. As in Section 2, the homogeneous time is $\tau = \mu \int_0^t T^{d/2} \, dt'$, the only change being that T is given by (5.5). In (5.2) we have $a(\tau) = 0$ and the homogeneous solution is the one studied in Section 3. All the results of Sections 3 and 4 apply here.

For both spatial force and velocity force plus source $a_1 d + a_2 = 0$ let us choose the same T and the same i.c. for $f(\boldsymbol{\eta}^2, 0)$. Both inhomogeneous distributions are identical for any t value. All the solutions of Section 4 (spatial forces) can be reinterpreted here. We only have to deduce the force $a_1(t) = \nu_0 T^{1/2} + \frac{1}{2}\partial_t \log T$. Note also that if $T(\infty) = \mathrm{const}$, then $a_1(\infty) = \nu_0 T(\infty)^{1/2} > 0$ (assuming $\partial_t T \to 0$) and $a_2 < 0$. This means that the absolute Maxwellians are obtained with an asymptotic source term (not a sink). Consequently, the reduced distributions of Figs. 1 and 2 and the asymptotic ocillting Maxwellians of Fig. 3 are numerical examples of distributions corresponding to $da_1 + a_2 = 0$.

5.5. Local Entropy

We come back to the general intermolecular forces and assume both external velocity force and source term as in (5.1). We define the same local density and current entropy as in Section 2.5 and find $\mathscr{I}_{0t} + \partial_{\mathbf{X}} \cdot \boldsymbol{\mathscr{I}} + a_2 \mathscr{I}_0 \geqslant 0$, where the nonnegative rhs is still $-\int d\mathbf{v} \log f \, \mathrm{Col}\, f$. Now we assume even inhomogeneous similarity distributions and again obtain $\mathscr{I}_0 = \mathscr{I}_0(t)$, $\mathscr{I} = \langle \mathbf{v} \rangle \mathscr{I}_0(t)$. From Section 5.2 we have $\partial_{\mathbf{X}} \cdot \langle \mathbf{v} \rangle = da_1 - \frac{1}{2} d\partial_t \log T$, whence

$$\partial_t \left[\mathscr{I}_0 T^{-d/2} \exp \int_0^t (a_1 d + a_2)\, dt' \right] \geqslant 0$$

and for the particular mixing $da_1 + a_2 = 0$, as in Section 2 for the spatial force, we find that $\mathscr{I}_0 T^{-d/2}$ is a nondecreasing function of the time.

6. CONCLUSION

The aim of the paper was to show that the previously found[(4)] BKW inhomogeneous similarity distribution of the B.E. with well-defined external forces is not the only one leading toward absolute Maxwellian equilibrium states. Here, from the homogeneous distributions associated with the inhomogeneous similarity ones, we have explicitly constructed a whole class of such distributions. The external forces determine entirely the time dependence of both the temperature and the density of the inhomogeneous distribution as well as the time of the associated homogeneous one. The absolute Maxwellian relaxation is obtained from the asymptotic behavior of the temperature. It follows that this property is independent of both the intermolecular forces and of the nature of the homogeneous distribution. However, at present, only for a Maxwellian interaction and spatially homogeneous distributions with even velocity dependence do we completely know the methods for the explicit construction of distributions. Consequently, we have restricted our study to these cases, although, for instance,[5] the formalism could be extended to velocity-dependent spatially homogeneous distributions. The Laguerre moments of the Laguerre series which build up the distributions are deduced both from the time dependence of the outside force and the i.c.

For the inhomogeneous similarity distributions without external forces, the Nikolskii result holds.[(1)] The distribution corresponds to a gas in expansion: the temperature, the density, and the distribution go to zero when the time goes to infinity. One might think that this is due to the construction of very peculiar classes of inhomogeneous distributions. However, in the infinite medium considered here, with neither boundary conditions nor external forces, it may be that the expansion of the gas holds for a

[5] We note the existence of different homogeneous formalisms which do not share even velocity distributions and for which the present framework could be applied.[(19)]

larger class of solutions. Physically it seems difficult for the gas to be confined (equivalently, a Maxwellian relaxation) if nothing prevents its expansion in space.

In Appendix A we recall that for time- and space-dependent solutions of the B.E. without collision term or external forces,[13] T, ρ, f vanish at infinity. Let us assume a weaker condition: for t larger than some fixed t_0 (arbitrarily large, but finite), the collision term is negligible. Then the asymptotic behavior for t larger than t_0 is provided by the linear B.E. part and the previous expansion result holds. Also note that in the one-dimensional Kac model, for the explicit inhomogeneous distribution relaxing toward a Maxwellian,[13] in order to maintain positivity in the whole space, it is necessary to introduce elstic walls, sources, and sinks.

For the class of linear velocity-dependent force plus source term, $a_1 d + a_2 = 0$, it is clear that both contribute to the existence of absolute Maxwellians because this happens for a particular mixing. Do these forces and source term have a physical meaning? Linear velocity forces seem similar to viscosity. On the one hand, they restrain the motion of the particules. On the other hand, in the B.E. they modify the macroscopic conservation laws (mass,...). Then, the particular mixing with the source term is such that they disappear in the mass conservation law, $(\partial_t + \partial_{\mathbf{X}} \cdot \langle \mathbf{v} \rangle)\rho = -(a_1 d + a_2)\rho = 0$. There is some kind of dissipation, which is balanced by the source term. A similar cancellation occurs in the inequality for the derivative of the local entropy (see Section 5.5). For the spatially dependent forces, the resulting potential is the time-dependent harmonic potential.

Recall that in another context (distributions for vanishing collision term), Boltzmann[12] introduced these potentials. Although in the literature they are considered with some disdain, the fact that they reappear here naturally in the search for Maxwellian states suggests that they are important in the theory of the B.E. *Further, here their time dependence is such that they vanish when $t \to \infty$; nevertheless, the distribution remains confined within an absolute Maxwellian. They play a role similar to a cutoff or a screen, necessary to avoïd the trivial expansion result.*

As a by-product of these inhomogeneous similarity solutions with Maxwellian equilibrium states, we note that the Tjon[8] overshoot effect appears in a more complicated way than in the homogeneous energy dependent formalism. Not only is the i.c. important, but so are the microscopic cross section as well as the sign and the decrease of the outside potential. Some of these new features were already present[20] in the one-dimensional Kac model.

APPENDIX A. $\mathscr{L} f_{\mathrm{LM}}=0$, $\mathscr{L}\eta^2=0$

A1. Spatial Forces $\mathbf{A}(\mathbf{x}, t)$ (see also Refs. 4 and 13)

Let us write $f_{\mathrm{LM}}=\nu(\mathbf{x}, t)\exp(-\eta^2/2)$, $\eta=\gamma(\mathbf{v}-\langle\mathbf{v}\rangle)$, $\nu=\rho(2\pi T)^{-d/2}$; we assume $\mathscr{L} f_{\mathrm{LM}}=0$ and study the constraints on $T=\gamma^{-2}$, ν, $\langle\mathbf{v}\rangle$, $\mathbf{A}$. We have

$$(\partial_t+\mathbf{v}\cdot\partial_{\mathbf{X}})\log\nu=\mathscr{L}\eta^2/2, \qquad \mathscr{L}=\partial_t+\mathbf{v}\cdot\partial_{\mathbf{X}}\partial_{\mathbf{X}}+\mathbf{A}(\mathbf{x}, t)\cdot\partial_{\mathbf{V}} \tag{A1}$$

[note that (A1) becomes $\mathscr{L}\eta^2=0$ if $\nu=\text{const}$]. The lhs is a first-order polynomial in the components v_i (of $\mathbf{v}$), while the rhs is of third order: $\mathbf{v}^2 v_i$ leads to $\partial_{\mathbf{X}}\gamma=0$ or $T=T(t)$ and v_i^2, $v_i v_j$ give

$$\langle\mathbf{v}\rangle_i=\alpha_i(t)-1/2(T_t/T)x_i+\sum_j\omega_{ij}(t)x_j, \qquad \omega_{ij}+\omega_{ji}=0, \qquad \omega_{ii}=0 \tag{A2}$$

Equating the coefficients of the remaining v_i and constant terms in (A1),

$$\gamma^{-1}\,\partial_{x_i}\log\nu=\partial_t(\gamma\langle\mathbf{v}\rangle_i)+\sum_j\gamma\langle\mathbf{v}\rangle_j\,\omega_{ij}-A_i\gamma \tag{A3}$$

$$\partial_t\Lambda=-2\langle\mathbf{v}\rangle\cdot\mathbf{A}/T, \qquad \Lambda=\log\nu^2-\langle\mathbf{v}\rangle^2/T \tag{A4}$$

where $\langle\mathbf{v}\rangle_i$ and A_i are the components of $\langle\mathbf{v}\rangle$ and $\mathbf{A}$. [Notice that (A3)–(A4) lead to $(\partial_t+\langle\mathbf{v}\rangle\cdot\partial_{\mathbf{X}})\log\nu=0$]. We substitute (A2) into (A3),

$$\begin{aligned}-\gamma^{-1}\,\partial_{x_i}&\log\nu+A_i\gamma\\ &=\left\{\partial_t(\alpha_i\gamma)+\gamma\sum\alpha_j\omega_{ij}\right\}+\left\{x_i\left(\gamma_{tt}-\sum_j\gamma\omega_{ij}^2\right)\right\}\\ &\quad+\left\{\sum_j x_j\left[\gamma^{-1}\partial_t\omega_{ij}T^{-1}+\sum_{l,i\neq j}\omega_{il}\omega_{lj}\gamma\right]\right\}\end{aligned} \tag{A3$'$}$$

and remark that the rhs is of first order in the spatial x_i components. Applying ∂_{x_j} and $i\leftrightarrow j$, we find

$$A_{i,x_j}-A_{j,x_i}=2T\partial_t T^{-1}\omega_{ij}, \qquad 2\partial^2_{x_i x_j}\Lambda=(A_{i,x_j}+A_{j,x_i})T^{-1} \tag{A3$''$}$$

From (A3$''$) we see that the force $\mathbf{A}(\mathbf{x}, t)$ is conservative if $T^{-1}\omega_{ij}=\text{const}$ and nonconservative otherwise. Finally, we rewrite (A3) with Λ defined in (A4):

$$\partial_{\mathbf{X}}\Lambda+2\partial_t\langle\mathbf{v}\rangle/T=2\mathbf{A}/T \tag{A3bis}$$

A1.1. From $\mathscr{L}f_{\rm LM}=0$ and $\mathrm{A}=0$ It Follows That $f_{\rm LM}\to 0$ and $T\to 0$. From (A3″) we find both $T^{-1}\omega_{ij}=\text{const}$ and Λ of the type $\lambda_0(t)+\sum \lambda_{1,i}(t)x_i+\lambda_{2,i}(t)x_i^2$. Applying (A3bis) or

$$\partial_t \Lambda=\partial_{x_i}\Lambda+\partial_{t^2}T^{-1}\langle \mathbf{v}\rangle_i=0$$

$$\text{to } \Lambda \text{ and } \langle \mathbf{v}\rangle_i, \text{ we find } \partial^2_{t^2}T^{-1}\alpha_i=\partial^3_{t^3}T^{-1}=0$$

and for the general solution

$$\begin{aligned}
T^{-1}&=C_0+C_1t+C_2t^2\\
T^{-1}\langle \mathbf{v}\rangle_i&=\bar{\alpha}_i t+\bar{\bar{\alpha}}_i+(C_1/2+C_2t)x_i+\sum \bar{\omega}_{ij}x_j\\
\log v^2&=T^{-1}\langle \mathbf{v}\rangle^2+\alpha_0-2\sum \bar{\alpha}_i x_i+C_2\mathbf{x}^2
\end{aligned}\tag{A5}$$

where C_0, C_1, C_2, $\bar{\alpha}_i$, $\bar{\bar{\alpha}}_i$, $\bar{\omega}_{ij}$, and α_0 are constants and $T^{-1}\alpha_i=\bar{\bar{\alpha}}_i+\bar{\alpha}_i t$. The important point is that T abd $f_{\rm LM}\to 0$ when $t\to\infty$. We can choose either $\bar{\omega}_{ij}=0$ or $\bar{\omega}_{ij}\neq 0$.

A1.2. $\mathrm{A}=0$ and $\mathrm{v}=\text{const}$ or Equivalently $\mathscr{L}\eta^2=0$ without Force:

(i) $\omega_{ij}=0$. From (A2′) we find $\gamma_{tt}=0$, $d_i\gamma=\bar{\alpha}_i=\text{const}$,

$$T^{-1/2}=\gamma=C_0+C_1t, \qquad \gamma\langle \mathbf{v}\rangle=\boldsymbol{\alpha}+C_1\mathbf{x}\tag{A6}$$

where C_0 and C_1 are constants and still T, $f_{\rm LM}\to 0$ when $t\to\infty$.

(ii) $\omega_{ij}\neq 0$. On the one hand, each of the three braces of (A3′) is zero. On the other hand, T and $\langle \mathbf{v}\rangle$ are given by the first two relations (A5), with C_i, $\bar{\alpha}_i$, $\bar{\bar{\alpha}}_i$, $\bar{\omega}_{ij}$ constrained by (A4)–(A5). From (A3′) we get

$$\begin{aligned}
&\sum_l \omega_{il}\omega_{lj}=0, \qquad i\neq j\\
&\sum_j \left(\frac{\omega_{ij}}{T}\right)^2=\sum \bar{\omega}_{ij}^2=\omega^2(0), \qquad i \text{ independent}
\end{aligned}\tag{A7}$$

The first relation gives constraints on the antisymmetric ω_{ij} that are trivial for $d=2$, impossible for $d=3$, and satisfied for $d=4$ if we choose $\omega_{24}=-\varphi\omega_{13}$, $\omega_{23}=\varphi\omega_{14}$, $\omega_{34}=\varphi\omega_{12}$, and $\varphi^2=1$. Equations (A4)–(A5) can be expressed by $\partial_t T^{-1}\langle \mathbf{v}\rangle^2=0$:

$$\begin{aligned}
&C_0C_2-\tfrac{1}{4}C_1^2=\omega^2(0), \qquad \sum (C_1\bar{\alpha}_i^2-2\bar{\alpha}_i\bar{\bar{\alpha}}_iC_2)=0\\
&\sum (C_0\bar{\alpha}_i^2-C_2\bar{\bar{\alpha}}_i^2)=0, \qquad \sum C_i\bar{\bar{\alpha}}_i^2-2C_0\bar{\alpha}_i\bar{\bar{\alpha}}_i=0\\
&C_1\bar{\alpha}_i-C_2\bar{\bar{\alpha}}_i+2\sum \bar{\omega}_{ij}\bar{\alpha}_j=0, \qquad -C_1\bar{\bar{\alpha}}_i+2C_0\bar{\alpha}_i+2\sum \bar{\omega}_{ij}\bar{\bar{\alpha}}_j=0
\end{aligned}\tag{A8}$$

The first relation defines $\omega(0)$. For the others we can choose $\bar{\alpha}_i = \bar{\bar{\alpha}}_i = 0$, otherwise, for instance, the last two relations give for $d = 2$

$$C_1 D = 2(\bar{\alpha}_2 \bar{\bar{\alpha}}_2 + \bar{\alpha}_1 \bar{\bar{\alpha}}_1), \qquad C_2 D = \bar{\alpha}_1^2 + \bar{\alpha}_2^2$$
$$C_0 D = \bar{\bar{\alpha}}_1^2 + \bar{\bar{\alpha}}_2^2, \qquad D = (\bar{\alpha}_2 \bar{\bar{\alpha}}_1 - \bar{\alpha}_1 \bar{\bar{\alpha}}_2)\, \bar{\omega}_{12}^{-1} \tag{A9}$$

while all other relations are identities. In conclusion, we still have $T, f_{\mathrm{LM}} \to 0$ when $t \to \infty$.

1.3. The Assumption $v = v(t)$ Leads Necessarily to $\mathscr{L}\boldsymbol{\eta}^2 = 0$ and $v = \text{const}$. In (A3) both the lhs and the rhs are zero. We sum $\sum \langle \mathbf{v} \rangle_i (\gamma^{-1}\gamma_t \langle \mathbf{v} \rangle_i - A_i)$ and find $-\sum \langle \mathbf{v} \rangle_i \langle \mathbf{v} \rangle_j \omega_{ij}$ from the rhs of (A3). Due to the antisymmetry of ω_{ij}, this result is zero, and substituting in (A4), we find $v = \text{const}$ and necessarily $\mathscr{L}\boldsymbol{\eta}^2 = 0$. Returning to the explicit expression (A3[1]), the rhs defines the different parts of the i-th component force, its x_i-dependent and -independent parts, and its x_j ($j \neq i$)-dependent part. Let us call $a(t)\mathbf{x}$ the part of Astrictly proportional to $\mathbf{A}$; then

$$(T^{-1/2})_{tt} = a(t)\, T^{-1/2} \tag{A10}$$

with $a(t)$ arbitrary. If $a(t)t^2 \to 0$ when $t \to \infty$, there exist solutions $T \to \text{const}$ and $f_{\mathrm{LM}} \to f_{\text{Abs Maxw}} = \exp(-\text{const} \times \mathbf{c}^2/2)$.

A2. Linear Velocity Force Plus Source Term $a_1(t)\, \partial_{\mathbf{V}} \cdot \mathbf{v} + a_2(t)$

We start with

$$\mathscr{L} = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{X}} + a_1(t)\, \partial_{\mathbf{V}} \cdot \mathbf{v} + a_2(t) + \mathbf{A}(t) \cdot \partial_{\mathbf{V}}$$

Define $\bar{\mathscr{L}} = \mathscr{L} - d a_1 + a_2$, the differential part of $\mathscr{L}$, and seek the constraints on T, v, $\langle \mathbf{v} \rangle$, $\mathbf{A}$, a_1, a_2 due to $\mathscr{L} f_{\mathrm{LM}} = \mathscr{L}(v \exp(-\boldsymbol{\eta}^2/2) = 0$, $\boldsymbol{\eta} = \gamma(\mathbf{v} - \langle \mathbf{v} \rangle)$, $\gamma^2 = T^{-1}$:

$$a_1 d + a_2 + (\partial_t + \mathbf{v} \cdot \partial_{\mathbf{X}}) \log v = \mathscr{L}\boldsymbol{\eta}^2/2$$
$$\bar{\mathscr{L}} = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{X}} + [a_1 \mathbf{v} + \mathbf{A}(t)] \cdot \partial_{\mathbf{V}} \tag{A1'}$$

The lhs is still third order in v_i and the rhs first order; (A2) becomes

$$\langle \mathbf{v} \rangle_i = \alpha_i + [a_1(t) - 1/2 T_t T^{-1}] x_i + \sum \omega_{ij} x_j, \qquad \omega_{ij} + \omega_{ji} = 0 \tag{A2'}$$

It turns out that (A3) is the same, while (A4) changes:

$$2(a_1 d + a_2) + \partial_t A = 2T^{-1} \langle \mathbf{v} \rangle \cdot \mathbf{A} \tag{A4'}$$

with the same A as in (A4). Equation (A3′) is slightly modified; we add $x_i\partial_t(a_1\gamma)$ and the last brace on the rhs has a supplementary term:

$$\left\{\sum_j x_j\left[\gamma^{-1}\partial_t T^{-1}\omega_{ij}+\gamma a_1\omega_{ij}+\sum_l \omega_{il}\omega_{lj}\right]\right\},\qquad i\neq j \qquad \text{(A3a')}$$

A2. If we Assume $a_1d+a_2=0$, $\mathbf{v}=\mathbf{v}(t)$, Then Necessarily $\mathbf{v}=$const and $\mathscr{L}\eta^2=0$. Equation (A1′) gives $\partial_t \log v=\mathscr{L}\eta^2/2$. From (A4′), which becomes identical to (A4), we have $\partial_t \log v=\sum\langle\mathbf{v}\rangle_i(\partial_t\gamma\langle\mathbf{v}\rangle_i-A_i\gamma)$, and is equal to $-\sum\langle\mathbf{v}\rangle_i\langle\mathbf{v}\rangle_j\omega_{ij}=0$.

A22. Assuming $\mathscr{L}\eta^2=0$ or $\mathscr{L}(\eta_j)=0$, We Find $\mathbf{v}=\mathbf{v}(t)$. From (A1′) we have $a_1d+a_2+(\partial_t+\mathbf{V}\cdot\partial_{\mathbf{X}})\log v=0$; since v is $\mathbf{v}$ independent, it follows that

$$v(t)=v(0)\exp-\int_0^t (a_1d+a_2)\,dt' \qquad \text{(A11)}$$

A2.3. Study of $\mathscr{L}\eta^2=0$. The lhs of (A1′) is zero and either $a_1d+a_2=0$ and $v=$const, or $a_1d+a_2\neq 0$ and v is given in (A11). For the study of $\mathscr{L}\eta^2=0$, the term a_1d+a disappears in (A3), (A4′), (A3a′). We first show that (A4′) is a consequence of (A3) or (A3a′). It is sufficient to note that

$$\sum\langle\mathbf{v}\rangle_i(\partial_t\gamma\langle\mathbf{v}\rangle_i-A_i\gamma)=-\sum\langle\mathbf{v}\rangle_i\langle\mathbf{v}\rangle_j\omega_{ij}=0$$

Consequently, we reduce our study to (A2′)–(A3) or equivalently (A3a′). In (A3a′) the lhs depends only on t, while the rhs has x_i and x_j terms; it follows that

$$A_i\gamma=\partial_t(\alpha_i\gamma)+\sum\alpha_j\gamma\omega_{ij},\qquad \gamma_{tt}=\sum_j\gamma\omega_{ij}^2-(a_1\gamma)_t$$

independent of i (A12)

$$\partial_t\log\gamma^2\omega_{ij}+a_1=0,\qquad \sum_l\omega_{il}\omega_{lj}=0,\qquad i\neq j$$

(i) $\omega_{ij}=0$. We find $\langle\mathbf{v}\rangle=\boldsymbol{\alpha}+v_0T^{1/2}\mathbf{x}$ with $a_1=(v_0-\gamma_t)\gamma^{-1}$, ($v_0=$const.) leading to

$$\gamma=T^{-1/2}=\left[\exp\left(-\int_0^t a_1\,dt'\right)\right]\left(1+v_0\int_0^t\exp\int_0^{t'}a_1\,dt''\,dt'\right)$$

For instance, if $a_1 = \text{const} > 0$, then

$$\gamma = \gamma(\infty) + e^{-a_1 t}[1 - \gamma(\infty)] \to \gamma(\infty) = v_0 a_1^{-1} \tag{A13}$$

Even if $T \to \text{const}$, $f_{\text{LM}} \to f_{\text{Abs Max}}$ only if $a_1 d + a_2 = 0$, which means $a_2 = \text{const} < 0$ in the previous example (or the presence of a source term a_2).

(ii) $\omega_{ij} \neq 0$. In (A12), the second and the fourth relations cannot be satisfied for $d = 3$. There is no problem for $d = 2$, while for $d = 4$ we can choose $\omega_{24} = -\varphi\omega_{13}$, $\omega_{23} = \varphi\omega_{14}$, $\omega_{34} = \varphi\omega_{12}$, and $\varphi^2 = 1$. For the first relation of (A12) we can either choose $\alpha_i = A_i = 0$ or consider that they are arbitrary functions linked by this relation. The second and third relations give

$$\gamma^2 \omega_{ij} = \left[\exp\left(-\int_0^t a_1\, dt'\right)\right] \omega_{ij}(0)$$

$$\partial_t(\gamma a_1) + \gamma_{tt} - \gamma^{-3}\left[\exp\left(-2\int_0^t a_1\, dt'\right)\omega^2(0) = 0 \tag{A14}$$

$$\omega^2(0) = \sum_j \omega_{ij}^2(0)$$

leading to a nonlinear differential equation for T (or γ), once a_1 is given.

APPENDIX B

B1. Solitons and Bisolitons of the Ernst–Hendriks Integrodifferential Nonlinear Equation

We have, with q not necessarily an integer,

$$\mathscr{L}\left(H\left(\frac{p}{p+1}, \tau\right)\right) = \mathscr{N}(H^2(r)), \qquad \mathscr{L} = 1 + \partial_\tau$$
$$\mathscr{N} = \left(\frac{1+p}{p}\right)^q \frac{\Gamma(2q)}{(\Gamma(q))^2} \int_0^{p/(p+1)} dr \left[r\left(1 - \frac{r(p+1)}{p}\right)\right]^{q-1} \tag{B1}$$

We easily find

$$\mathscr{N}(r^n) = \left(\frac{p}{p+1}\right)^n \frac{\Gamma(2q)\,\Gamma(q+n)}{\Gamma(q)\,\Gamma(2q+n)}$$

B1.1. "Solitons." We start with an ansatz $H = (1 + \lambda z)\, {}_1F_0(a;\ ; z)$, where $z = ve^{\rho\tau}p/(p+1)$ and ρ, λ, v, and a are unknown constants. We recall that ${}_1F_0((a;\ ;z))^2 = {}_1F_0(2a;\ ;z)$ and substitute the ansatz into (B1):

$$\mathscr{L}(H) = \frac{1}{T(a)} \sum \frac{\Gamma(a+n-1)}{n!} z^n (n\rho+1)[a-1+n(\lambda+1)]$$

$$\mathscr{N}(H^2) = \frac{\Gamma(2q)}{\Gamma(2a)\,\Gamma(q)} \sum \frac{\Gamma(2a-2+n)}{n!} \frac{\Gamma(n+q)}{\Gamma(n+2q)} z^n P_{22}(n,a) \tag{B2}$$

$$P_{22}(n,a) = [n(1+\lambda)]^2 + n[4a-3+4\lambda(a-1)-\lambda^2] + (2a-1)(2a-2)$$

In $\mathscr{L}(H)$ and $\mathscr{N}(H^2)$, from the asymptotic n dependence of the coefficients of z^n, we find $a = q+1$; it follows that

$$2(2q+1)(n\rho+1)[n(1+\lambda)+q] = P_{22}(n, q+1)$$

From the equality of the coefficients of n^2, n, const we find $2(2q+1)\rho = 1+\lambda = -q$ and finally, with v an arbitrary constant,

$$H = \left[1-(q+1)\frac{p}{p+1} v e^{\rho\tau}\right] {}_1F_0\left(q+1;\ ;\frac{vp}{p+1} e^{\rho\tau}\right) \tag{B3}$$

B12. "Bisolitons." We start with an ansatz $H = (1+\lambda z+\mu y)$ ${}_1F_0(a;\ ;z)$, where

$$z = \varphi \frac{p}{p+1}(1+ve^{\rho\tau}), \qquad y = \varphi \frac{p}{p+1} ve^{\rho\tau}$$

and a, λ, μ, v, φ, and ρ are unknown constants. We substitute the ansatz into (B1) and find

$$\mathscr{L}(H) = \frac{1}{T(a)} \sum \frac{T(a+n-1)}{n!} \left(\frac{\varphi p}{p+1}\right)^n (1+ve^{\rho\tau})^{n-2} Q_2(n, a, e^{\rho\tau}) \tag{B4a}$$

$$Q_2(n,a,e^{\rho\tau}) = (ve^{\rho\tau}+1)[a-1+n(1+\lambda+\mu)][1+ve^{\rho\tau}(n\rho+1)] - \mu n\{1+ve^{\rho\tau}[1+\rho(n-1)]\}$$

$$\mathscr{N}(H^2) = \frac{T(2q)}{T(2a)\,T(q)} \sum \frac{T(2a-2+n)\,\Gamma(n+q)}{n!\,T(n+2q)} P_2(n,a,e^{\rho\tau}) \times \left(\frac{\varphi p}{p+1}\right)^n (1+ve^{\rho\tau})^{n-2} \tag{B4b}$$

$$P_2 = (1+ve^{\rho\tau})^2 P_{22}(n,a) + n(n-1)\, ve^{\rho\tau}[\mu^2 ve^{\rho\tau} + 2\lambda\mu(1+ve^{\rho\tau})] + 2\mu n(2a+n-2)\, ve^{\rho\tau}(1+ve^{\rho\tau})$$

P_{22} was defined in (B2). From the asymptotic n dependence we still find $a = q+1$ and consequently

$$2(2q+1)\, Q_2(n, q+1, e^{\rho\tau}) = P_2(n, q+1, e^{\rho\tau}) \tag{B5}$$

Q_2 and P_2 can be written $Q_2 = v^2 e^{2\rho\tau} Q_{22} + v e^{\rho\tau} Q_{21} + Q_{20}$ and $P_2 = v^2 e^{2\rho\tau} \bar{P}_{22} + v e^{\rho\tau} \bar{P}_{21} + \bar{P}_{20}$. Relation (5) holds for each of the couples $(Q_{22}, \bar{P}_{22})$ $(Q_{21}, \bar{P}_{21})$, $(Q_{20}, \bar{P}_{20})$. Each couple leads to a second-order polynomial in n and so to three relations. From the first couple we find $2\rho(2q+1) = 1 + \lambda + \mu = -q$, from the second, $\lambda = -1$ or $\mu = -q$, and identities from the third. Finally, v and φ are arbitrary constants; the solutions depend on two variables: $\varphi p/(p+1)$, where we define $v_2 = \varphi$, and $\varphi v e^{\rho\tau} p/(p+1)$, where we define $v_1 = \varphi v$:

$$H = \left[1 - \frac{v_2 p}{p+1} - (q+1)\frac{v_1 p e^{\rho\tau}}{p+1}\right] {}_1F_0\left(q+1;\quad ;\frac{v_2 p}{p+1} + \frac{v_1 p e^{\rho\tau}}{p+1}\right),$$

$$\rho = \frac{-q}{2(2q+1)} \tag{B6}$$

which gives back the "solitons" (B4) if $v_2 = 0$.

B1.3. Boltzmann Distributions Associated with the Solitons and Bisolitons. $F(u, \tau)$ is defined by

$$H(p, \tau) = (1+p)^q [\Gamma(d/2)]^{-1} \int_0^\infty du\, u^{d/2-1}\, {}_1F_1\left(q, \frac{d}{2}, -pu\right) F(u, \tau) \tag{B7}$$

and we must invert (B7). As an intermdiate step, we find the solution as a Laguerre series. Using

$${}_1F_0(a;\quad ; x) = [\Gamma(a)]^{-1} \sum \Gamma(a+n) x^n/n!$$

and

$$\frac{p^n}{(p+1)^{n+q}} \frac{\Gamma(n+q)}{n!} \frac{\Gamma(d/2)}{1(q)}$$
$$= \int_0^\infty e^{-u} u^{d/2-1}\, {}_1F_1(q, d/2, -pu)\, L_n^{d/2-1}(u)\, du \tag{B8}$$

we find for the bisolitons (B6):

$$F(u, \tau) = e^{-u} \sum z^n [1 - n(1 - v_2/z)] L_n^{d/2-1}(u), \qquad z = v_2 + v_1 e^{\rho\tau} \tag{B9}$$

Note that v_2 is the coefficient of $e^{-u}L_1^{d/2-1}(u)$. In the soliton case $v_2=0$, this coefficient vanishes, whereas in the bisoliton case, it is independent of time. In order to sum up the series, we use

$$\begin{aligned} &\sum L_n^{d/2-1}(u)\, z^n(1-z)^{d/2} = e^{uz/(z-1)} \\ &z\,\partial_z \sum L_n^{(d/2-1)}(u) z^n = (1-z)^{-(d/2+1)} z e^{uz/(z-1)} L_1^{d/2-1}[u/(1-z)] \end{aligned} \tag{B10}$$

with $L_1^{(\alpha)}(u)=\alpha+1-u$, and finally find

$$F(u,\tau)=(1-z)^{d/2}e^{-u/(1-z)}\left[1+\frac{z-v_2}{1-z}\left(\frac{u}{1-z}-\frac{d}{2}\right)\right] \tag{B11}$$

z and ρ are defined in (B6)–(B9). For the "solitons," $v_2=0$, and in (B9) the Laguerre moments $z^n(1-n)$ vanish for $n=1$.

B2. Tjon–Wu NL PDE: Fundamental Solutions and Weierstrassian Solution

We restrict attention to $q=1$ in (B1) or

$$(1+\partial_\tau)(1+\partial_x)\, H(x=\log[p/(p+1)],\tau)=H^2$$

(i) The fundamental solutions are of the type $H=1+wP(w)^{p+1}$, $w=\exp[N(x+\rho\tau)]$, N integer >1, and satisfy the equation

$$(N+1)(\beta P_w+\alpha w P_{ww})=P^2$$

$$\rho=\frac{1-N}{N(N+1)},\qquad \beta=-2N^2+3N+1,\qquad \alpha=N(N+1)$$

No known closed solution not violating physical constraints was obtained, while numerical calculations suggest that they could violate positivity.

(ii) The equianharmonic Weierstrass solution is a mixing of the two above solutions for $N=2,3$. Let us define $H=1-w^2P(w)$, $w=\exp(x+\rho\tau)$; then

$$(6\rho+1)(P+P_w)+w^2\rho P_{ww}+w^2P^2=0$$

Choosing $\rho=-1/6$, we find

$$P_{ww}=6P^2,\qquad H=1-w^2\mathscr{P}(w+\alpha,\, g_2=0,\, g_3),\qquad w=\frac{p}{p+1}e^{-\tau/6} \tag{B12}$$

α and g_3 are arbitrary constants. We follow the notations of Southard[17] and prove that that the associated $F(u, \tau)$ Boltzmann distribution violates positivity. We know that

$$\mathscr{P} = y^{-2}\left(1 + \sum e_n y^{6n}\right), \qquad y = w + \alpha, \qquad e_1 = g_3/8$$

$$e_n(n-1)(6n+1) = \sum e_m e_{m'}, \qquad m + m' = n$$

In the degenerate case, $g_3 = 0$, $H = 1 - w^2/(1+w)^2$ for $\alpha = 1$, which is nothing but the BKW soliton (B3) for $q = 1$. Then $g_3 \neq 0$ corresponds to an extension of the BKW solution.

First we show that for $w > 0$ finite, $H(w, \tau)$ necessarily violates positivity. $\mathscr{P}$, an even y function, is a doubly periodic function with an infinite number of poles, with half-periods w_2 and w_2':

$$w_2 = \int_{l_2}^{\infty} dv\, S(v)^{-1/2}, \qquad w_2' = i \int_{-\infty}^{l_2} dv\, |S(v)|^{-1/2}$$

$$l_2 = \operatorname{sign} g_3 |g/4|^{1/3}, \qquad S = 4v^3 - g_3$$

It is sufficient to study a fundamental period parallelogram (FPP). For $g_3 > 0$, $w + \alpha \in [2Mw_2,\ 2(M+1)w_2]$, $\mathscr{P} > 0$ has values symmetric with respect to $(2M+1)w_2$, M integer; $\mathscr{P}$ goes to infinity at the edges of the interval and is positive at the minimum $\mathscr{P}((2M+1)w_2) = l_2 > 0$. It follows that H and $-w^2\mathscr{P}$ go to minus infinity an infinite number of times when w is along the real axis. For $g_3 < 0$, we use the relation $\mathscr{P}(y, g_2 = 0, g_3) = -\mathscr{P}(iy, g_2 = 0, -g_3)$, $-g_3 > 0$. In the FFP we look for $g_3 > 0$ at the diagonals parallel to the imaginary axis. Then $\mathscr{P}(iy, g_2 = 0, -g_3)$ along the interval $[2Mw_2', (2M+2)w_2']$ is real, symmetric with respect to $(2M+1)w_2'$, goes to minus infinity at $2Mw_2'$, $(2M+2)w_2'$ has two well-defined zeros, becomes positive between these two zeros and is maximum at w_2', where $\mathscr{P}$ has the value $l_2 > 0$. It follows that $H = 1 + \omega^2\mathscr{P}(iy, g_2 = 0, -g_3 > 0)$ has an infinite number of negative subintervals. These is no continuity between the BKW $g_3 = 0$ case and the Weierstrass solution, because in the complex y plane we have one pole in the first case and an infinite number in the other.

Second, we prove that necessarily $F(u, \tau)$ violates positivity if H does. This is trivial for $d = 2$, because the transform $F \to H$ becomes the Laplace transform

$$H(p, \tau) = (1 + p) \int_0^{\infty} F(u, \tau)\, e^{-pu}\, du$$

and $F>0$ leads to $H>0$. In fact, for $q>0$ and $d>2$, the kernel ${}_1F_1(q, d/2, -pu)$ of the transform (B7) has no negative[21] $-pu$ zeros; whence $F>0$ still requires $H>0$. Here, $q=1$ is a particular case of this general result.

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